A model of **Presburger Arithmetic**, $(\Gamma, +, <, 0, 1)$
abelian group with least positive element 1 satisfy
schema

$$\forall x \in \Gamma\, \exists y \in \Gamma\, \exists i \in \{0, 1, \ldots, n-1\}(x = ny$$

Definition 1. *$\widehat{\mathbb{Z}}$ is the group of sequences $(r_n)_{n\in}$*

$$\begin{array}{ll} 0 \le r_n < n & \textit{for all } n \ge 1, \\ r_{nm} \equiv r_n \pmod{n} & \textit{for all } n, m \ge 1, \end{array}$$

where

$$(r_n)_{n\in\mathbb{N}} + (s_n)_{n\in\mathbb{N}} = (r_n + s_n \pmod{n}))$$

Definition 2. *For* Γ *a Presburger group, the* **na**
map

$$\varrho\colon \Gamma \to \widehat{\mathbb{Z}}$$

is the homomorphism

$$\varrho(\gamma) = (\gamma \pmod 1), \gamma \pmod 2), \gamma \pmod 3)$$

for all $\gamma \in \Gamma$*.*

Definition 3. Γ *is* **divisible** *if for all* $x \in \Gamma$ *and al*
there is $y \in \Gamma$ *with* $ny = x$*.*

Proposition 4. *If* Γ *is a Presburger group then*
convex subgroup, so $\Gamma/\mathbb{Z}$ *is divisible.*

Theorem 5. *Suppose* $\overline{\Gamma}$ *is a divisible ordered abe*
$\overline{\varrho}\colon \overline{\Gamma} \to \widehat{\mathbb{Z}}/\mathbb{Z}$ *a homomorphism. Then for some Pre*
Γ *we have*

$$\begin{array}{ccccc}
\Gamma & \longrightarrow & \Gamma/\mathbb{Z} & \xrightarrow{\theta} & \overline{\Gamma} \\
\downarrow \varrho & & & & \downarrow \overline{\varrho} \\
\widehat{\mathbb{Z}} & & \longrightarrow & & \widehat{\mathbb{Z}}/\mathbb{Z}
\end{array}$$

where $\theta\colon \Gamma/\mathbb{Z} \to \overline{\Gamma}$ *is an isomorphism and the un*
represent natural quotient maps.

Notation 6. *We write* $\overline{\varrho}\colon \overline{\Gamma} \to \widehat{\mathbb{Z}}/\mathbb{Z}$ *to mean the i*

$$\varrho/\mathbb{Z}\colon \Gamma/\mathbb{Z} \to \widehat{\mathbb{Z}}/\mathbb{Z}.$$

Definition 7. *Let* $\alpha\colon \overline{\Gamma} \to \overline{\Gamma}$ *be an automorphis* *ordered divisible abelian group so that*

$$\overline{\varrho}(\alpha(a)) = \overline{\varrho}(a) \qquad \text{for all } a \in \overline{\Gamma}.$$

Then we say the map $\alpha\colon \overline{\Gamma} \to \overline{\Gamma}$ **preserves residu** **residue-automorphism**.

Proposition 8. *If* $\alpha\colon \overline{\Gamma} \to \overline{\Gamma}$ *is a residue-automor* *lifts to an automorphism:*

$$\widehat{\alpha}\colon \Gamma \to \Gamma.$$

Definition 9. Γ *is* **homogeneous** *if for* $\bar{a}, \bar{b} \in \Gamma^n$
$\mathrm{tp}(\bar{b})$ *there is some* $\alpha \in \mathrm{Aut}(\Gamma)$ *with* $\bar{a}\alpha = \bar{b}$.

Definition 10. *For* $a, b \in \overline{\Gamma}$ *with* $a, b > \mathbb{Z}$, *we defin*

$$\mathrm{st}\left(\frac{a}{b}\right) = \{\, q \in \mathbb{Q} : qb < a \}\,.$$

This is an extended cut, identified with an exte
$[0, \infty] \subseteq \mathbb{R} \cup \{\infty\}$, *where* $r = \sup \mathrm{st}\left(\frac{a}{b}\right)$.

Lemma 11. *For* $a, b, c \in \overline{\Gamma}, q \in \mathbb{Q}$ *the following ho*

1. $\mathrm{st}\left(\frac{a}{b}\right) \cdot \mathrm{st}\left(\frac{b}{c}\right) = \mathrm{st}\left(\frac{a}{c}\right)$ *provided the LHS is defined;*

2. $\mathrm{st}\left(\frac{qa}{b}\right) = q \cdot \mathrm{st}\left(\frac{a}{b}\right)$*;*

3. $\mathrm{st}\left(\frac{a}{qb}\right) = \frac{1}{q} \cdot \mathrm{st}\left(\frac{a}{b}\right)$ *for* $q \neq 0$*;*

4. $\mathrm{st}\left(\frac{a+b}{c}\right) = \mathrm{st}\left(\frac{a}{c}\right) + \mathrm{st}\left(\frac{b}{c}\right)$ *provided the RHS is defi*

5. *if* $a \leq b$ *then* $\mathrm{st}\left(\frac{a}{c}\right) \leq \mathrm{st}\left(\frac{b}{c}\right)$*;*

6. *if* $\mathrm{st}\left(\frac{a}{b}\right) \notin \{0, \pm\infty\}$ *then* $\mathrm{st}\left(\frac{a}{b}\right) = \mathrm{st}\left(\frac{b}{a}\right)^{-1}$.

Definition 12. *If* $a, b \in \overline{\Gamma}$ *then* $a \equiv b$ *if either* $a = b$
and

$$\mathrm{st}\left(\frac{a}{b}\right) \notin \{0, \pm\infty\}.$$

This is an equivalence relation so we may define:

Definition 13. $\mathsf{V} = \overline{\Gamma}/\equiv$ *is the set of* **values** *of* $\overline{\Gamma}$
ordered by

$$a/\equiv\, < b/\equiv \quad\Longleftrightarrow\quad a/\equiv\, \neq b/\equiv \text{ and } |a| <$$

The **valuation map** $\mathsf{v}\colon \overline{\Gamma} \to \mathsf{V}$ *is defined by*

$$a \mapsto a/\equiv.$$

$\mathsf{v}\colon \overline{\Gamma} \to \mathsf{V}$ has the following properties:

1. $\mathsf{v}(qa) = \mathsf{v}(a)$ for all $q \in \mathbb{Q} \setminus \{0\}$;

2. if $|a| \leq |b|$ then $\mathsf{v}(a) \leq \mathsf{v}(b)$;

3. if $n|a| < |b|$ for all $n \in \mathbb{N}$ then $\mathsf{v}(a) < \mathsf{v}(b)$;

4. $\mathrm{st}\left(\frac{a}{b}\right) = 1 \quad \Rightarrow \quad \mathsf{v}(a-b) < \mathsf{v}(a), \mathsf{v}(b)$;

5. $\mathrm{st}\left(\frac{a}{b}\right) \neq 1 \quad \Rightarrow \quad \mathsf{v}(a-b) = \max(\mathsf{v}(a), \mathsf{v}(b))$.

Definition 14. *The set* $B \subseteq \overline{\Gamma}$ *is* **strongly indepe**
and every nontrivial $\mathbb{Q}$*-linear combination*

$$a = q_1 b_1 + \cdots + q_n b_n$$

has value

$$\mathsf{v}(a) = \max\{\, \mathsf{v}(b_j) : 1 \leq j \leq n, q_j \neq 0\}$$

where $\bar{q} \in \mathbb{Q}$ *and* $\bar{b} \in B$.

Lemma 15 (Exchange Lemma). *If* $a_1, \ldots, a_n$ *a*
dependent in $\overline{\Gamma}$, *and* $a \in \overline{\Gamma}$ *then*

either $a \in \langle a_1, \ldots, a_n \rangle$
or $\exists\, a_{n+1} \in \langle a_1, \ldots, a_n, a \rangle$ *such that* $a_1, \ldots, a$
are strongly independent and $a \in \Big\langle a_1, .$

Lemma 16. *If* $\{x_1, \ldots, x_n\} \subseteq \overline{\Gamma}$ *and* $\{y_1, \ldots, y_n\}$
strongly independent sets, then the following are

1. $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$;

2. $\varrho(x_i) = \varrho(y_i)$ *and* $\mathrm{st}\left(\frac{x_i}{x_j}\right) = \mathrm{st}\left(\frac{y_i}{y_j}\right)$ *for all* $1 \leq i \leq j$

Proof. Follows from quantifier elimination.

Theorem 17. *Suppose* Γ *is 2-homogeneous, the*
are equivalent:

1. Γ *has no smallest non-standard value, and there*
   trivial $g \in G$;

2. *there is some* $x \in \Gamma$ *with* $\varrho(x) = 0$ *and there are*
   elements with value less than $\mathsf{v}(x)$;

3. *there is a value-defying automorphism* $h \in G$;

4. Γ *contains a unique maximal convex submodel wit*
   ing a dense linear order, with $\varrho^{-1}(r)$ *dense in* Γ
   so that for all non-standard $x, y, z \in \Gamma$ *there is* w
   $\mathrm{st}\left(\frac{w}{z}\right) = \mathrm{st}\left(\frac{x}{y}\right)$.

Definition 18. *A Presburger group* Γ *is* **pseuc**
saturated *if* $\Gamma \ncong \mathbb{Z}$ *and*

1. *for* $\overline{\varrho}\colon \overline{\Gamma} \to \widehat{\mathbb{Z}}/\mathbb{Z}$ *and each* $\mathbb{Z}+r \in \operatorname{Im}(\overline{\varrho})$*, the inverse*
$r)$ *is dense in* $\overline{\Gamma}$*;*

2. *for* $x, y, z \in \Gamma$ *with* $z \notin \mathbb{Z}$*, there is some* $w \notin \mathbb{Z}$ *for v*
$$\operatorname{st}\left(\frac{w}{z}\right) = \operatorname{st}\left(\frac{x}{y}\right);$$

3. *the set of values* V *is a dense linear order with*
having least point 0 *and no greatest point.*

Theorem 19. *Let* $\overline{\Gamma}$ *be prs and suppose that* $(a_1$
$\overline{\Gamma}$ *and* $(b_1, \ldots, b_n) = \overline{b} \in \overline{\Gamma}$ *are such that* $\mathrm{tp}(\bar{a}) =$
there are strongly independent sets $\{a'_1, \ldots, a'_n\}$ *a*
for which:

1. $\left\langle a_1, \ldots, a_n \right\rangle = \left\langle a'_1, \ldots, a'_n \right\rangle$ *and* $\left\langle b_1, \ldots, b_n \right\rangle = \left\langle b'_1, \ldots\right.$

2. $a_i = q_1 a'_1 + \cdots + q_n a'_n$ *if and only if* $b_i = q_1 b'_1 + \cdots$
$q_1, \ldots, q_n \in \mathbb{Q}$*;*

3. $\overline{\varrho}(a'_i) = \overline{\varrho}(b'_i)$ *for* $1 \leq i \leq n$*;*

4. $\mathrm{st}\left(\frac{a'_i}{a'_j}\right) = \mathrm{st}\left(\frac{b'_i}{b'_j}\right)$ *for* $1 \leq i \leq j \leq n$.

Theorem 20. *Suppose Γ is countable prs, and t*

$$A = \{a_1, a_2, \ldots, a_n\}, \qquad B = \{b_1, b_2, \ldots$$

are strongly independent subsets of Γ with:

1. $\varrho(a_i) = \varrho(b_i)$ *for all* $1 \leq i \leq n$;

2. $\operatorname{st}\left(\frac{a_i}{a_j}\right) = \operatorname{st}\left(\frac{b_i}{b_j}\right)$ *for all* $1 \leq i, j \leq n$.

Then there exists an automorphism $\theta \colon \Gamma \to \Gamma$ ma for all i.

Corollary 21. *If Γ is countable prs then Γ is hom*

Example 22.

$$G_{\mathsf{v}} = \{g \in G : \mathsf{v}(\gamma g) = \mathsf{v}(\gamma) \textit{ for all } \gamma \in \Gamma$$

is a non-trivial, proper, closed normal subgroup o

Theorem 23. *Let* Γ *be countable prs with* $h \in G$
$a_1, \ldots, a_n \in \Gamma \backslash \mathbb{Z}$ *and* $b_1, \ldots, b_n \in \Gamma \backslash \mathbb{Z}$ *are such that*
Suppose further that $\infty \in \mathrm{stQ}(\langle h \rangle)$. *Then there*
such that

$$\bar{a} h^{g_1} h^{-g_2} = \bar{b}.$$

Lemma 24. *Suppose* $h \in G$ *preserves values, an*
arbitrary. Then $\mathsf{v}(\gamma g^{-1} h g) = \mathsf{v}(\gamma)$ *for all* $\gamma \in \Gamma$.

Definition 25. *If* $S_n \subseteq (\mathrm{stQ}(\overline{\Gamma}))^n \subseteq \left(\mathbb{R}^*_{>0}\right)^n$ *anc*
then the stQ-**closure** *properties are as follows:*

1. *Each* S_n *is nonempty and closed under pointwise*

2. *each* S_n *is closed under pointwise inversion;*

3. *if* $(r_1, \ldots, r_n) \in S$ *and* $m \leq n$ *then*

$$(r_1, \ldots, r_{m-1}, r_{m+$$

4. *if* $(r_1, \ldots, r_n) \in S$ *and* $m \leq n + 1$ *then there exist*
r'_m *so that* $(r_1, \ldots, r_{m-1}, r'_m, r_m, \ldots, r_n) \in S$*.*

Definition 26. *If* $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\overline{\Gamma}))^n$ *is* stQ-*close*
the set of residue automorphisms

$$G_S^{<\omega} = \Bigg\{ g \in G_{\mathsf{v}} : \forall\, n \in \omega\, \forall\, \mathsf{v}(x_1) < \cdots < \mathsf{v}(x_n)$$

$$\left(\mathsf{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathsf{st}\right.$$

Theorem 27. *If* $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\overline{\Gamma}))^n$ *is* stQ-*close*
a closed normal subgroup of G*.*

Theorem 28. *Suppose $N \trianglelefteq G$ is a normal subgr*
automorphisms. If

$$S = \left\{ \left(\mathsf{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathsf{st}\left(\frac{x_n g}{x_n}\right) \right) : n \in \omega, g \in N, \mathsf{v}(x_1)$$

then S satisfies the stQ*-closure properties and N*

Theorem 29. *Suppose that G has trivial centre*
a closed normal subgroup of G. If

$$S = \left\{ \left(\mathsf{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathsf{st}\left(\frac{x_n g}{x_n}\right) \right) : n \in \omega, g \in N, \mathsf{v}(x_1)$$

then $N = G_S^{<\omega}$.

Definition 30. *Let* $\gamma_1, \gamma_2 \in \Gamma$. *We say* γ_1 *is* **close**

$$\mathrm{st}\left(\frac{\gamma_1}{\gamma_2}\right) = 1 \quad \textit{or} \quad \gamma_1 = \gamma_2 = 0.$$

We denote this property by writing $\gamma_1 \frown \gamma_2$.

Proposition 31. *Suppose* $h \in G$ *preserves values.*
ther that $a_1, \ldots, a_n \in \overline{\Gamma}$ *are such that* $0 < \mathrm{v}(a_n) < \cdot$
that $b_1, \ldots, b_n \in \overline{\Gamma}$ *are such that* $\mathrm{tp}(a_i) = \mathrm{tp}(b_i)$
$\mathrm{stQ}(h)$ *for* $1 \leq i \leq n$. *Then there is some* $w \in \langle$
$a_i w \frown b_i$ *for all* $1 \leq i \leq n$.

Proposition 32. *Suppose $h \in G$ is non-trivial, fix segment and that $\gamma h \frown \gamma$ for all $\gamma \in \overline{\Gamma}$. Suppos $a_1 < \cdots < a_n \in \overline{\Gamma}$ are strongly independent, that that $b_k \frown a_k$ so that $\overline{\varrho}(b_k) = \overline{\varrho}(a_k)$. Then ther $g_k \in \left\langle h^G \right\rangle$ which fixes each a_i with $\mathsf{v}(a_i) \geq \mathsf{v}(a_k)$ ex maps a_k to b_k and so that $\gamma g_k \frown \gamma$ for all $\gamma \in \overline{\Gamma}$.*

Corollary 33. *Suppose $h \in G$ is non-trivial, fixe segment and that $\gamma h \frown \gamma$ for all $\gamma \in \overline{\Gamma}$. Suppose a*

$$\{a_1, \ldots, a_n\} \quad \text{and} \quad \{b_1, \ldots, b_n\}$$

are both strongly independent, that $a_i \frown b_i$ with and $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$ for all $1 \leq i, j \leq n$. Then the $w \in \left\langle h^G \right\rangle$ which maps $w\colon a_i \mapsto b_i$ for all $1 \leq i \leq n$.

Proposition 34. *If* $T_1, T_2 \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\overline{\Gamma}))^n$ *are s*
there exists some $(r_1, \ldots, r_n)$ *with*

$$(r_1, \ldots, r_n) \in T_1 \quad \textit{but} \quad (r_1, \ldots, r_n) \notin$$

then there is a residue automorphism $g \in G$ *wit*
$g \notin G_{T_2}^{<\omega}$.

Proposition 35. *Suppose* $T_1 \subseteq T_2$. *Then* $G_{T_1}^{<\omega} \subseteq$

Proposition 36. *Suppose T_1 and T_2 are both* stQ
$\langle T_1 \cup T_2 \rangle = \{ t_1.t_2 : t_1 \in T_1, t_2 \in T_2 \}$ *is* stQ*-closed.*

Definition 37. *If $S \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\overline{\Gamma}))^n$ then the s*
of S is defined to be:

$$\overline{S}^{\mathrm{stQ}} = \bigcup_{\substack{T\subseteq\langle S\rangle \\ T \text{ stQ-}\textit{closed}}} T.$$

Proposition 38. *If $S \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\overline{\Gamma}))^n$ then $\overline{S}^{\mathrm{stQ}}$*
and $\overline{S}^{\mathrm{stQ}} \subseteq \langle S \rangle$.

Proposition 39. *Let* T_1 *and* T_2 *be* stQ*-closed.* T

$$G^{<\omega}_{\langle T_1 \cup T_2 \rangle} = \overline{\left\langle G^{<\omega}_{T_1} \cup G^{<\omega}_{T_2} \right\rangle}.$$

Proposition 40. *Let* T_1 *and* T_2 *be* stQ*-closed.* T

$$G^{<\omega}_{\overline{T_1 \cap T_2}^{\mathrm{stQ}}} = G^{<\omega}_{T_1} \cap G^{<\omega}_{T_2}.$$

Definition 41. *Let* $\Gamma_{\mathsf{v}} \subseteq \bigcup_{n\in\omega}\Gamma^n$ *be the set of*
$\mathsf{v}(x_1) < \cdots < \mathsf{v}(x_n)$.

Definition 42. *If* T *is* stQ*-closed and* $\bar{x}, \bar{y} \in \overline{\Gamma_{\mathsf{v}}}$ *we*
$\bar{y}$ *if* $\mathsf{tp}(\bar{x}) = \mathsf{tp}(\bar{y})$ *and*

$$\left(\mathsf{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathsf{st}\left(\frac{x_n}{y_n}\right)\right) \in T.$$

Lemma 43. *Suppose* $\bar{x}, \bar{y} \in \Gamma_{\mathsf{v}}$. *Then* $\bar{x} \sim_T \bar{y}$
$\bar{x}g = \bar{y}$ *for some* $g \in G_T^{<\omega}$.

Theorem 44. *In the diagram:*

1. $G_{T_1}^{<\omega} \subseteq G_{T_2}^{<\omega}$ *if and only if for all* $\bar{x}, \bar{y} \in \Gamma_{\mathsf{V}}$ *we have* $\bar{x} \sim_{T_2} \bar{y}$.

2. *The arrows are bijections.*

Proposition 45. *Suppose $\sim$ is a G-invariant equiv*
on Γ_{V} and that

1. *if $x_1, \ldots, x_n \sim y_1, \ldots, y_n$ and $m \leq n$ then*

$$x_1, \cdots, x_{m-1}, x_{m+1}, \cdots, x_n \sim y_1, \cdots, y_{m-1}, y_{m+}$$

2. *if $x_1, \ldots, x_n \sim y_1, \ldots, y_n$ and $m \leq n + 1$ then there*
pair x'_m, y'_m with

$$x_1, \cdots, x_{m-1}, x'_m, x_m, \cdots, x_n \sim y_1, \cdots, y_{m-1}, y'_m,$$

3. *suppose $\bar{x}, \bar{y}$ are such that $\bar{x} \frown \bar{y}$, then $\bar{x} \sim \bar{y}$.*

Then there is an stQ*-closed T with $\bar{x} \sim \bar{y}$ if and o*