A Galois Correspondence for $\mathbb{Z}$-Groups

The $\mathbb{Z}$-Group Axioms

All of the $\mathbb{Z}$-groups discussed on this poster will be considered to be countable.

A **$\mathbb{Z}$-group**, $(\Gamma, +, <, 0, 1)$, is an ordered abelian group with least positive element 1 satisfying

$$\forall x \in \Gamma \, \exists y \in \Gamma \, \exists i \in \{0, 1, \dots, n-1\}(x = ny + i).$$

For every $n \in \mathbb{N}$. This axiom schema can be taken to mean that $\Gamma/\mathbb{Z}$ is divisible.

The ordering of a $\mathbb{Z}$-group is linear, discrete and respected by $+$. The standard example of a $\mathbb{Z}$-group is **the integers** with addition as the operation. In fact every $\mathbb{Z}$-group contains a copy of the integers as a convex subgroup.

$\mathbb{Z}$-groups are also often referred to as **Presburger groups**, a name which relates to Mojżesz Presburger, the twentieth century mathematician who first showed the completeness of the theory of $\mathbb{Z}$-groups in 1929 [3].

Figure 1 shows a pictorial representation of what a $\mathbb{Z}$-group might look like, whilst figure 2 shows a pictorial representation of what Mojżesz Presburger might have looked like.

Because of the completeness of the theory and its connection with the arithmetic of integers, $\mathbb{Z}$-groups are often considered in relation to automated theorem proving.

General Definitions

- The axiom given in the definition of a $\mathbb{Z}$-group allows us to define the **residue map** $\varrho\colon \Gamma \to \widehat{\mathbb{Z}}$. That is, the map
$$\varrho(\gamma) = (\gamma \pmod 1, \gamma \pmod 2, \gamma \pmod 3, \dots).$$
- Although we can't compute fractions $\frac{a}{b}$ in Γ, we can approximate the ratios between elements using real numbers:
$$\mathrm{st}\left(\frac{a}{b}\right) = \left\{ q \in \mathbb{Q} : \exists r, s \in \mathbb{N}, s > 0, \frac{r}{s} > q \text{ and } rb < sa \right\}.$$
This can be identified with the extended real $r = \sup \mathrm{st}\left(\frac{a}{b}\right) \in \mathbb{R} \cup \{\infty\}$.
- We call $\mathrm{stQ}(\Gamma)$ the set of **standard parts** achievable in Γ;
$$\mathrm{stQ}(\Gamma) = \left\{ \mathrm{st}\left(\frac{a}{b}\right) \in \mathbb{R} \cup \{\pm\infty\} : a, b \in \Gamma \right\}.$$
- Similarly we let $\mathrm{Res}(\Gamma)$ be the set of residues of Γ, *i.e.* the image of the natural map $\varrho\colon \Gamma \to \widehat{\mathbb{Z}}$.
- We can define an equivalence relation $a \equiv b$ for $a, b \in \Gamma$ if $a = b = 0$ or $a, b \neq 0$ with $\mathrm{st}\left(\frac{a}{b}\right) \notin \{0, \pm\infty\}$.
- This gives us the set of equivalence classes $\mathrm{V} = \Gamma/\equiv$ which are linearly ordered by
$$a/\equiv \, < b/\equiv \quad \iff \quad a/\equiv \, \neq b/\equiv \text{ and } |a| < |b|.$$
The **valuation map** $\mathrm{v}\colon \Gamma \to \mathrm{V}$ is defined by $a \mapsto a/\equiv$.
- We can define a notion of independence. We say that $B \subseteq \Gamma \setminus \mathbb{Z}$ is **strongly independent** if every nontrivial $\mathbb{Q}$-linear combination
$$a = q_1 b_1 + \dots + q_n b_n$$
has value
$$\mathrm{v}(a) = \max\{\, \mathrm{v}(b_j) : 1 \le j \le n, q_j \neq 0\}$$
where $\bar{q} \in \mathbb{Q}$ and $\bar{b} \in B$.

Figure 2: Mojżesz Presburger.

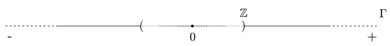

Figure 1: The general form of a $\mathbb{Z}$-group.

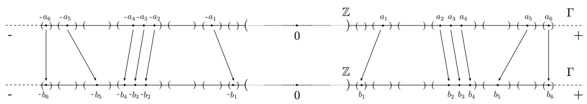

Figure 3: An automorphism moving strongly independent elements.

Automorphisms

Let $G = \mathrm{Aut}(\Gamma)$, the automorphism group of Γ. We want to know more about the structure of G and on this poster we look at the closed normal subgroups of G. We also set $G_{\mathrm{v}} = \{\, g \in G : \mathrm{v}(\gamma g) = \mathrm{v}(\gamma) \text{ for all } \gamma \in \Gamma\}$, the group of value-preserving automorphisms.

The important feature of **pseudo-recursively saturated** $\mathbb{Z}$-groups (see the box below left) is that they are **homogeneous** and so have abundant automorphisms. This is what the next theorem tells us.

Theorem 1. Suppose Γ is a pseudo-recursively saturated $\mathbb{Z}$-group, and that we have strongly independent subsets of Γ

$$A = \{a_1, a_2, \dots, a_n\}, \qquad B = \{b_1, b_2, \dots, b_n\},$$

which satisfy the following:

1. $\varrho(a_i) = \varrho(b_i)$ for all $1 \le i \le n$;
2. $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$ for all $1 \le i, j \le n$.

Then there is an automorphism $g \in G$ mapping a_i to b_i for all $1 \le i \le n$.

Figure 3 to the left indicates how such an automorphism might act.

For strongly independent sets in a pseudo-recursively saturated $\mathbb{Z}$-group, points 1 and 2 above are equivalent to saying $\mathrm{tp}(A) = \mathrm{tp}(B)$.

Pseudo-Recursive Saturation and Homogeneity

Looking at the automorphisms of $\mathbb{Z}$-groups, it is natural to look at a particular subclass of the $\mathbb{Z}$-groups, as is demonstrated by the next theorem.

Theorem 2. Suppose that Γ is a 2-homogeneous $\mathbb{Z}$-group, then the following are equivalent:

1. Γ has no smallest non-zero value, and there is a non-trivial automorphism $g\colon \Gamma \to \Gamma$;
2. there is some $x \in \Gamma$ such that $\varrho(x) = 0$ and there are non-standard elements with value less than that of x;
3. there is some value-defying automorphism $g\colon \Gamma \to \Gamma$;
4. Γ contains a unique maximal convex submodel with values forming a dense linear order, with $\varrho^{-1}(r)$ dense in Γ for all $r \in \mathrm{Res}(\Gamma)$ and so that for all non-standard $x, y, z \in \Gamma$ there exists some $w \in \Gamma$ such that $\mathrm{st}\left(\frac{w}{z}\right) = \mathrm{st}\left(\frac{x}{y}\right)$.

The unique maximal convex submodel described in part 4 above provides the type of $\mathbb{Z}$-group which turns out to be particularly interesting. We call a $\mathbb{Z}$-group which satisfies the properties given **pseudo-recursively saturated**. The definition is given in full to the right. The importance of pseudo-recursively saturated $\mathbb{Z}$-groups is hinted at by the following result and its corollary.

Proposition 3. Any recursively saturated $\mathbb{Z}$-group is pseudo-recursively saturated.

Corollary 4. If Γ is a pseudo-recursively saturated $\mathbb{Z}$-group then Γ is homogeneous.

Definition 5. A $\mathbb{Z}$-group Γ is **pseudo-recursively saturated** if $\Gamma \not\cong \mathbb{Z}$ and

- For $r \in \mathrm{Res}(\Gamma)$ the inverse image $\varrho^{-1}(r)$ is dense in Γ (in the sense of $<$);
- for $x, y, z \in \Gamma$ with $z \notin \mathbb{Z}$, there is some $w \notin \mathbb{Z}$ for which
$$\mathrm{st}\left(\frac{w}{z}\right) = \mathrm{st}\left(\frac{x}{y}\right);$$
- the set of values V is a dense linear order with respect to $<$ having least point 0 and no greatest point.

The notion of pseudo-recursive saturation was first used implicitly by Victor Harnik [1] in a different context. It was first made explicit by Richard Kaye [2] who has supervised this work.

Theorem 6. Suppose that Γ is a 1-homogeneous $\mathbb{Z}$-Group with values forming a dense linear order. Then the following are equivalent:

1. G/G_{v} is primitive on V;
2. Γ is pseudo-recursively saturated.

stQ-Closure

The following turns out to be instrumental in describing the closed normal subgroups of the pseudo-recursively saturated $\mathbb{Z}$-groups, and in giving the description of the Galois connection.

Definition 7. If $S_n \subseteq (\mathrm{stQ}(\Gamma))^n \subseteq (\mathbb{R}^*_{>0})^n$ and $S = \bigcup_{n\in\omega} S_n$ then the stQ**-closure** properties of the S_n are as follows:

- Each S_n is nonempty and closed under pointwise multiplication;
- each S_n is closed under inversion (where $(r_1, \dots, r_n)^{-1} = (r_1^{-1}, \dots, r_n^{-1})$);
- if $(r_1, \dots, r_n) \in S$ and $m \le n$ then $(r_1, \dots, r_{m-1}, r_{m+1}, \dots, r_n) \in S$;
- if $(r_1, \dots, r_n) \in S$ and $m \le n+1$ then there exists at least one r'_m so that $(r_1, \dots, r_{m-1}, r'_m, r_m, \dots, r_n) \in S$.

$G_S^{<\omega}$ and the Normal Subgroups

Definition 8. If $S \subseteq \cup_{n\in\omega}(\mathrm{stQ}(\Gamma))^n$ satisfies the stQ-closure properties from definition 7 above, then we define $G_S^{<\omega}$ to be the set of automorphisms

$$G_S^{<\omega} = \left\{ g \in G_{\mathrm{v}} : \begin{array}{c} \forall n \in \omega \, \forall \, \mathrm{v}(x_1) < \dots < \mathrm{v}(x_n) \\ \left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \dots, \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right) \in S \end{array} \right\}.$$

It turns out that these subgroups represent **all of the closed normal subgroups** of G, the automorphism group of Γ.

Theorem 9. Suppose Γ is a pseudo-recursively saturated $\mathbb{Z}$-group and $S \subseteq \cup_{n\in\omega}(\mathrm{stQ}(\Gamma))^n$. Then

1. $G_S^{<\omega}$ is a closed normal subgroup of G;
2. if G has trivial centre, then every closed normal subgroup of G is of the form $G_S^{<\omega}$ for some $S \subseteq \bigcup_{n\in\omega}(\mathbb{R}^*)^n$.

Note that G can only have trivial centre if $\mathrm{Res}(\Gamma) = \{0\}$.

Using these facts we are able to discover the pair of Galois connections given to the right.

A Pair of Galois Connections

Definition 10. Suppose $S_1 \subseteq \cup_{n\in\omega}(\mathrm{stQ}(\Gamma))^n$ is stQ-closed and $\bar{x}, \bar{y} \in \widetilde{\Gamma}^n$ with $\mathrm{v}(x_1) < \dots < \mathrm{v}(x_n)$. Then we say that $\bar{x} \sim_{S_1} \bar{y}$ if $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ and

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \dots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) \in S_1.$$

This is an equivalence relation by the stQ-closure conditions on S_1. For $\mathrm{v}(x_1) < \dots < \mathrm{v}(x_n)$ an equivalent definition is to say that $\bar{x} \sim_{S_1} \bar{y}$ if and only if $\bar{x}g = \bar{y}$ for some $g \in G_{S_1}^{<\omega}$.

Using this equivalence relation we can find two Galois connections: between the closed normal subgroups of G and certain automorphism invariant equivalence relations on Γ/V, and between these equivalence relations and the stQ-closed subsets of $\cup_{n\in\omega}(\mathrm{stQ}(\Gamma))^n$. These are described by the figure (4) on the right. Note that **although the diagram suggests that the ordering by inclusion and implication is linear, it is in fact a partial ordering** (which turned out to be too tricky to draw!).

The theorem needed to show we have Galois connections is given below:

Theorem 11. Suppose Γ is a pseudo-recursively saturated $\mathbb{Z}$-group and that $S_1, S_2 \in \cup_{n\in\omega}(\mathrm{stQ}(\Gamma))^n$ are both stQ-closed. Then

1. $G_{S_1}^{<\omega} \subseteq G_{S_2}^{<\omega}$ if and only if for all $\mathrm{v}(x_1) < \dots < \mathrm{v}(x_n)$ we have $\bar{x} \sim_{S_1} \bar{y} \quad \Rightarrow \quad \bar{x} \sim_{S_2} \bar{y}$.
2. The arrows represented in figure 4 are bijections.

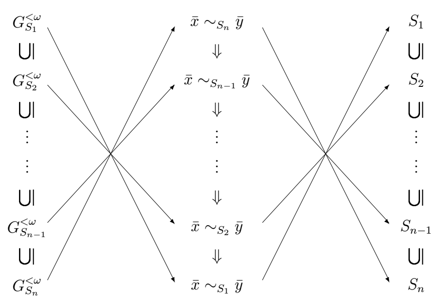

Figure 4: A Pair of Galois Connections.

References

[1] Victor Harnik. ω_1-like recursively saturated models of Presburger's arithmetic. *J. Symbolic Logic*, 51(2):421–429, 1986.

[2] Richard Kaye. Presburger arithmetic. Notes from a Birmingham University study group, 1997.

[3] Mojżesz Presburger. On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation. *Hist. Philos. Logic*, 12(2):225–233, 1991. Translated from the German and with commentaries by Dale Jacquette.