# Automorphisms of Models of Presburger Arithmetic

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A model of **Presburger Arithmetic**, $(\Gamma, +, <, 0, 1)$, is an ordered abelian group with least positive element 1 satisfying the axiom schema

$$\forall\, x \in \Gamma\, \exists\, y \in \Gamma\, \exists\, i \in \{0, 1, \ldots, n-1\}(x = ny + i)$$

**Definition 1.** *$\widehat{\mathbb{Z}}$ is the group of sequences $(r_n)_{n\in\mathbb{N}}$ such that*

$$\begin{array}{ll} 0 \leq r_n < n & \textit{for all } n \geq 1, \\ r_{nm} \equiv r_n \pmod{n} & \textit{for all } n, m \geq 1, \end{array}$$

*where*

$$(r_n)_{n\in\mathbb{N}} + (s_n)_{n\in\mathbb{N}} = (r_n + s_n \pmod{n})_{n\in\mathbb{N}}.$$

**Definition 2.** *For* $\Gamma$ *a Presburger group, the* **natural residue map**

$$\varrho\colon \Gamma \to \widehat{\mathbb{Z}}$$

*is the homomorphism*

$$\varrho(a) = (a \pmod{1}, a \pmod{2}, a \pmod{3}, \dots)$$

*for all* $a \in \Gamma$*.*

**Definition 3.** $\Gamma$ *is* **divisible** *if for all* $a \in \Gamma$ *and all* $n > 0$ *from* $\mathbb{N}$ *there is* $b \in \Gamma$ *with* $nb = a$*.*

**Proposition 4.** *If* $\Gamma$ *is a Presburger group then* $\mathbb{Z} \hookrightarrow \Gamma$ *as a convex subgroup, so* $\Gamma/\mathbb{Z}$ *is divisible.*

**Theorem 5.** *Suppose* $\widetilde{\Gamma}$ *is a divisible ordered abelian group and* $\widetilde{\varrho}\colon \widetilde{\Gamma} \to \widehat{\mathbb{Z}}/\mathbb{Z}$ *a homomorphism. Then for some Presburger group* $\Gamma$ *we have*

$$\begin{array}{ccccc} \Gamma & \longrightarrow & \Gamma/\mathbb{Z} & \xrightarrow{\theta} & \widetilde{\Gamma} \\ \downarrow \varrho & & & & \downarrow \widetilde{\varrho} \\ \widehat{\mathbb{Z}} & & \longrightarrow & & \widehat{\mathbb{Z}}/\mathbb{Z} \end{array}$$

*where* $\theta\colon \Gamma/\mathbb{Z} \to \widetilde{\Gamma}$ *is an isomorphism and the unmarked arrows represent natural quotient maps.*

**Notation 6.** *We write* $\widetilde{\varrho}\colon \widetilde{\Gamma} \to \widehat{\mathbb{Z}}/\mathbb{Z}$ *to mean the induced map*

$$\varrho/\mathbb{Z}\colon \Gamma/\mathbb{Z} \to \widehat{\mathbb{Z}}/\mathbb{Z}.$$

**Definition 7.** *Let* $\alpha\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ *be an automorphism of* $\widetilde{\Gamma}$ *as an ordered divisible abelian group so that*

$$\widetilde{\varrho}(\alpha(a)) = \widetilde{\varrho}(a) \qquad \text{for all } a \in \widetilde{\Gamma}.$$

*Then we say the map* $\alpha\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ **preserves residues** *and call* $\alpha$ *a* **residueautomorphism**.

**Proposition 8.** *If* $\alpha\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ *is a residueautomorphism, then* $\alpha$ *lifts to an automorphism:*

$$\widehat{\alpha}\colon \Gamma \to \Gamma.$$

**Definition 9.** $\Gamma$ *is* **homogeneous** *if for* $\bar{a}, \bar{b} \in \Gamma^n$ *with* $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ *there is some* $\alpha \in \mathrm{Aut}(\Gamma)$ *with* $\bar{a}\alpha = \bar{b}$*.*

**Definition 10.** *For* $a, b \in \Gamma$ *with* $a, b > \mathbb{Z}$*, we define*

$$\mathrm{st}\left(\frac{a}{b}\right) = \left\{\frac{n}{m} \in \mathbb{Q} : nb < ma\right\}.$$

*This is an extended cut, identified with an extended real* $r \in [0, \infty] \subseteq \mathbb{R} \cup \{\infty\}$*, where* $r = \sup \mathrm{st}\left(\frac{a}{b}\right)$*.*

**Lemma 11.** *For* $a, b, c \in \Gamma/\mathbb{Z}, q \in \mathbb{Q}$ *the following hold:*

1. $\mathrm{st}\left(\frac{a}{b}\right) \cdot \mathrm{st}\left(\frac{b}{c}\right) = \mathrm{st}\left(\frac{a}{c}\right)$ *provided the LHS is defined;*

2. $\mathrm{st}\left(\frac{qa}{b}\right) = q \cdot \mathrm{st}\left(\frac{a}{b}\right)$;

3. $\mathrm{st}\left(\frac{a}{qb}\right) = \frac{1}{q} \cdot \mathrm{st}\left(\frac{a}{b}\right)$ *for* $q \neq 0$;

4. $\mathrm{st}\left(\frac{a+b}{c}\right) = \mathrm{st}\left(\frac{a}{c}\right) + \mathrm{st}\left(\frac{b}{c}\right)$ *provided the RHS is defined;*

5. *if* $c > 0$ *and* $0 \leq a \leq b$ *then* $\mathrm{st}\left(\frac{a}{c}\right) \leq \mathrm{st}\left(\frac{b}{c}\right)$;

6. *if* $\mathrm{st}\left(\frac{a}{b}\right) \notin \{0, \pm\infty\}$ *then* $\mathrm{st}\left(\frac{a}{b}\right) = \mathrm{st}\left(\frac{b}{a}\right)^{-1}$.

**Definition 12.** *If* $a, b \in \tilde{\Gamma}$ *then* $a \equiv b$ *if either* $a = b = 0$ *or* $a, b \neq 0$ *and*

$$\mathrm{st}\left(\frac{a}{b}\right) \notin \{0, \pm\infty\}.$$

This is an equivalence relation so we may define:

**Definition 13.** $\mathsf{V} = \tilde{\Gamma}/\equiv$ *is the set of* **values** *of* $\tilde{\Gamma}$ *and is linearly ordered by*

$$a/\equiv \, < b/\equiv \quad \Longleftrightarrow \quad a/\equiv \, \neq b/\equiv \text{ and } |a| < |b|.$$

*The* **valuation map** $\mathsf{v}\colon \tilde{\Gamma} \to \mathsf{V}$ *is defined by*

$$a \mapsto a/\equiv.$$

$\mathsf{v}\colon \widetilde{\Gamma} \to \mathsf{V}$ has the following properties:

1. $\mathsf{v}(qa) = \mathsf{v}(a)$ for all $q \in \mathbb{Q} \setminus \{0\}$;

2. if $|a| \leq |b|$ then $\mathsf{v}(a) \leq \mathsf{v}(b)$;

3. if $n|a| < |b|$ for all $n \in \mathbb{N}$ then $\mathsf{v}(a) < \mathsf{v}(b)$;

4. $\mathsf{v}(a+b) \leq \max(\mathsf{v}(a), \mathsf{v}(b))$ with equality unless $\mathsf{v}(a) = \mathsf{v}(b)$;

5. if $\mathsf{v}(a) = \mathsf{v}(b)$ then $\mathsf{v}(a+b) < \mathsf{v}(a), \mathsf{v}(b)$ for $\mathsf{st}\left(\frac{a}{b}\right) = -1$ and if $\mathsf{st}\left(\frac{a}{b}\right) \neq -1$ then $\mathsf{v}(a) = \mathsf{v}(b) = \mathsf{v}(a+b)$.

**Definition 14.** *The set* $B \subseteq \tilde{\Gamma}$ *is* **strongly independent** *if* $0 \notin B$ *and every nontrivial* $\mathbb{Q}$*linear combination*

$$a = q_1 b_1 + \cdots + q_n b_n$$

*has value*

$$\mathrm{v}(a) = \max\{\, \mathrm{v}(b_j) : 1 \leq j \leq n, q_j \neq 0\}$$

*where* $\bar{q} \in \mathbb{Q}$ *and* $\bar{b} \in B$.

**Lemma 15 (Exchange Lemma).** *If* $a_1, \ldots, a_n$ *are strongly independent in* $\tilde{\Gamma}$*, and* $a \in \tilde{\Gamma}$ *then*

*either* $a \in \langle a_1, \ldots, a_n \rangle$

*or* $\exists\, a_{n+1} \in \langle a_1, \ldots, a_n, a \rangle$ *such that* $a_1, \ldots, a_n, a_{n+1}$ *are strongly independent and* $a \in \left\langle a_1, \ldots, a_n, a_{n+1} \right\rangle$.

**Lemma 16.** *If* $\{a_1, \dots, a_n\} \subseteq \Gamma$ *and* $\{b_1, \dots, b_n\} \subseteq \Gamma$ *are both strongly independent sets, then the following are equivalent:*

1. $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$;

2. $\varrho(a_i) = \varrho(b_i)$ *and* $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$ *for all* $1 \leq i \leq j \leq n$.

*Proof.* Follows from quantifier elimination. □

**Theorem 17.** *Suppose* $\Gamma$ *is 2homogeneous, then the following are equivalent:*

1. $\Gamma$ *has no smallest nonstandard value, and there is some non trivial* $g \in G$*;*

2. *there is some* $a \in \Gamma$ *with* $\varrho(a) = 0$ *and there are nonstandard elements with value less than* $\mathsf{v}(a)$*;*

3. *there is a valuedefying automorphism* $h \colon \Gamma \to \Gamma$*;*

4. $\Gamma$ *contains a convex submodel* $\Gamma'$ *with values forming a dense linear order, with* $\widetilde{\varrho}^{-1}(r)$ *dense in* $\widetilde{\Gamma}'$ *for all* $r \in \mathrm{Res}(\widetilde{\Gamma}')$ *and so that for all nonstandard* $a, b, c \in \Gamma$ *there is* $d \in \Gamma$ *such that* $\mathrm{st}\left(\frac{d}{c}\right) = \mathrm{st}\left(\frac{a}{b}\right)$.

**Definition 18.** *A Presburger group* $\Gamma$ *is* **pseudorecursively saturated** *if* $\Gamma \ncong \mathbb{Z}$ *and*

1. *for* $\widetilde{\varrho}\colon \Gamma/\mathbb{Z} \to \widehat{\mathbb{Z}}/\mathbb{Z}$ *and each* $\mathbb{Z}+r \in \mathrm{Im}(\widetilde{\varrho})$*, the inverse image* $\widetilde{\varrho}^{-1}(\mathbb{Z}+r)$ *is dense in* $\widetilde{\Gamma}$*;*

2. *for* $a, b, c \in \Gamma$ *with* $c \notin \mathbb{Z}$*, there is some* $d \notin \mathbb{Z}$ *for which*
$$\mathrm{st}\left(\frac{d}{c}\right) = \mathrm{st}\left(\frac{a}{b}\right);$$

3. *the set of values* $\mathsf{V}$ *is a dense linear order with respect to* $<$ *having least point* $0$ *and no greatest point.*

**Theorem 19.** *Let* $\Gamma$ *be prs and suppose that* $(a_1, \ldots, a_n) = \bar{a} \in \widetilde{\Gamma}$ *and* $(b_1, \ldots, b_n) = \bar{b} \in \widetilde{\Gamma}$ *are such that* $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. *Then there are strongly independent sets* $\{a'_1, \ldots, a'_n\}$ *and* $\{b'_1, \ldots, b'_n\}$ *for which:*

1. $\left\langle a_1, \ldots, a_n \right\rangle = \left\langle a'_1, \ldots, a'_n \right\rangle$ *and* $\left\langle b_1, \ldots, b_n \right\rangle = \left\langle b'_1, \ldots, b'_n \right\rangle$;

2. $a_i = q_1 a'_1 + \cdots + q_n a'_n$ *if and only if* $b_i = q_1 b'_1 + \cdots + q_n b'_n$ *where* $q_1, \ldots, q_n \in \mathbb{Q}$;

3. $\widetilde{\varrho}(a'_i) = \widetilde{\varrho}(b'_i)$ *for* $1 \leq i \leq n$;

4. $\mathrm{st}\left(\frac{a'_i}{a'_j}\right) = \mathrm{st}\left(\frac{b'_i}{b'_j}\right)$ *for* $1 \leq i \leq j \leq n$.

**Theorem 20.** *Suppose $\Gamma$ is countable prs, and that*

$$A = \{a_1, a_2, \dots, a_n\}, \qquad B = \{b_1, b_2, \dots, b_n\},$$

*are strongly independent subsets of $\Gamma$ with:*

1. $\varrho(a_i) = \varrho(b_i)$ *for all* $1 \leq i \leq n$;

2. $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$ *for all* $1 \leq i, j \leq n$.

*Then there exists an automorphism $\theta\colon \Gamma \to \Gamma$ mapping $a_i$ to $b_i$ for all $i$.*

**Corollary 21.** *If $\Gamma$ is countable prs then $\Gamma$ is homogeneous.*

**Example 22.**

$$G_{\mathsf{v}} = \{g \in G : \mathsf{v}(ag) = \mathsf{v}(a) \text{ for all } a \in \Gamma\}$$

*is a nontrivial, proper, closed normal subgroup of $G$.*

**Theorem 23.** *Let $\Gamma$ be countable prs with $h \in G$ and suppose $a_1, \ldots, a_n \in \Gamma \setminus \mathbb{Z}$ and $b_1, \ldots, b_n \in \Gamma \setminus \mathbb{Z}$ are such that $\mathsf{tp}(\bar{a}) = \mathsf{tp}(\bar{b})$. Suppose further that $\infty \in \mathsf{stQ}(\langle h \rangle)$. Then there are $g_1, g_2 \in G$ such that*

$$\bar{a} h^{g_1} h^{-g_2} = \bar{b}.$$

**Lemma 24.** *Suppose $h \in G$ preserves values, and let $g \in G$ be arbitrary. Then $\mathsf{v}(ag^{-1}hg) = \mathsf{v}(a)$ for all $a \in \Gamma$.*

**Definition 25.** *If* $S_n \subseteq (\mathrm{stQ}(\tilde{\Gamma}))^n \subseteq \left(\mathbb{R}^*_{>0}\right)^n$ *and* $S = \bigcup_{n \in \omega} S_n$ *then the* stQ**closure** *properties are as follows:*

1. *Each* $S_n$ *is nonempty and closed under pointwise multiplica tion;*

2. *each* $S_n$ *is closed under pointwise inversion;*

3. *if* $(r_1, \ldots, r_n) \in S$ *and* $m \leq n$ *then*

$$(r_1, \ldots, r_{m-1}, r_{m+1}, \ldots, r_n) \in S;$$

4. *if* $(r_1, \ldots, r_n) \in S$ *and* $m \leq n+1$ *then there exists at least one* $r'_m$ *so that* $(r_1, \ldots, r_{m-1}, r'_m, r_m, \ldots, r_n) \in S$.

**Definition 26.** *If* $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\tilde{\Gamma}))^n$ *is* stQ*closed, then* $G_S$ *is the set of residue automorphisms*

$$G_S = \left\{ g \in G_{\mathsf{v}} : \forall\, n \in \omega\, \forall\, \mathsf{v}(a_1) < \cdots < \mathsf{v}(a_n) \right.$$
$$\left. \mathsf{st}\left(\frac{a_1 g}{a_1}\right), \ldots, \mathsf{st}\left(\frac{a_n g}{a_n}\right)\right) \in S \right\}.$$

**Theorem 27.** *If* $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\tilde{\Gamma}))^n$ *is* stQ*closed then* $G_S$ *is a closed normal subgroup of* $G$*.*

**Theorem 28.** *Suppose* $N \trianglelefteq G$ *is a normal subgroup of residue automorphisms. If*

$$S = \left\{ \mathsf{st}\left(\frac{a_1 g}{a_1}\right), \ldots, \mathsf{st}\left(\frac{a_n g}{a_n}\right)\right) : n \in \omega, g \in N, \mathsf{v}(a_1) < \cdots < \mathsf{v}(a_n) \right\}$$

*then* $S$ *satisfies the* stQ *closure properties and* $N \subseteq G_S$.

**Theorem 29.** *Suppose that* $G$ *has trivial centre and that* $N$ *is a closed normal subgroup of* $G$. *If*

$$S = \left\{ \mathsf{st}\left(\frac{a_1 g}{a_1}\right), \ldots, \mathsf{st}\left(\frac{a_n g}{a_n}\right)\right) : n \in \omega, g \in N, \mathsf{v}(a_1) < \cdots < \mathsf{v}(a_n) \right\}$$

*then* $N = G_S$.

**Definition 30.** *Let $a, b \in \Gamma$. We say $a$ is* **close to** *$b$ if*

$$\mathrm{st}\left(\frac{a}{b}\right) = 1 \qquad \text{or} \qquad a = b \in \mathbb{Z}.$$

*We denote this property by writing $a \frown b$.*

**Proposition 31.** *Suppose $h \in G$ preserves values. Suppose further that $a_1, \ldots, a_n \in \widetilde{\Gamma}$ are such that $0 < \mathrm{v}(a_n) < \cdots < \mathrm{v}(a_1)$ and that $b_1, \ldots, b_n \in \widetilde{\Gamma}$ are such that $\mathrm{tp}(a_i) = \mathrm{tp}(b_i)$ with $\mathrm{st}\left(\frac{b_i}{a_i}\right) \in \mathrm{stQ}(h)$ for $1 \leq i \leq n$. Then there is some $w \in \left\langle h^G \right\rangle$ such that $a_i w \frown b_i$ for all $1 \leq i \leq n$.*

**Proposition 32.** *Suppose $h \in G$ is nontrivial, fixes some initial segment and that $\gamma h \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$. Suppose further that $a_1 < \cdots < a_n \in \widetilde{\Gamma}$ are strongly independent, that $1 \leq k \leq n$ and that $b_k \frown a_k$ so that $\widetilde{\varrho}(b_k) = \widetilde{\varrho}(a_k)$. Then there exists some $g_k \in \left\langle h^G \right\rangle$ which fixes each $a_i$ with $\mathsf{v}(a_i) \geq \mathsf{v}(a_k)$ except $a_k$, which maps $a_k$ to $b_k$ and so that $\gamma g_k \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$.*

**Corollary 33.** *Suppose $h \in G$ is nontrivial, fixes some initial segment and that $\gamma h \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$. Suppose also that*

$$\{a_1, \ldots, a_n\} \quad \text{and} \quad \{b_1, \ldots, b_n\}$$

*are both strongly independent, that $a_i \frown b_i$ with $\widetilde{\varrho}(a_i) = \widetilde{\varrho}(b_i)$ and $\mathsf{st}\left(\frac{a_i}{a_j}\right) = \mathsf{st}\left(\frac{b_i}{b_j}\right)$ for all $1 \leq i, j \leq n$. Then there exists some $w \in \left\langle h^G \right\rangle$ which maps $w \colon a_i \mapsto b_i$ for all $1 \leq i \leq n$.*

**Proposition 34.** *If* $T_1, T_2 \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\tilde{\Gamma}))^n$ *are* stQ*closed and there exists some* $(r_1, \ldots, r_n)$ *with*

$$(r_1, \ldots, r_n) \in T_1 \quad \textit{but} \quad (r_1, \ldots, r_n) \notin T_2,$$

*then there is a residue automorphism* $g \in G$ *with* $g \in G_{T_1}$ *but* $g \notin G_{T_2}$.

**Proposition 35.** *Suppose* $T_1 \subseteq T_2$. *Then* $G_{T_1} \subseteq G_{T_2}$.

**Proposition 36.** *Suppose* $T_1$ *and* $T_2$ *are both* stQ *closed. Then* $\langle T_1 \cup T_2 \rangle = \{ t_1.t_2 : t_1 \in T_1, t_2 \in T_2\}$ *is* stQ *closed.*

**Definition 37.** *If* $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\tilde{\Gamma}))^n$ *then the* stQ **reduction** *of* $S$ *is defined to be:*

$$\overline{S}^{\mathsf{stQ}} = \bigcup_{\substack{T \subseteq \langle S \rangle \\ T \ \mathsf{stQ}\ \textit{closed}}} T.$$

**Proposition 38.** *If* $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\tilde{\Gamma}))^n$ *then* $\overline{S}^{\mathsf{stQ}}$ *is* stQ *closed and* $\overline{S}^{\mathsf{stQ}} \subseteq \langle S \rangle$.

**Proposition 39.** *Let* $T_1$ *and* $T_2$ *be* stQ *closed. Then*

$$G_{\langle T_1 \cup T_2 \rangle} = \overline{\left\langle G_{T_1} \cup G_{T_2} \right\rangle}.$$

**Proposition 40.** *Let* $T_1$ *and* $T_2$ *be* stQ *closed. Then*

$$G_{\overline{T_1 \cap T_2}^{\mathrm{stQ}}} = G_{T_1} \cap G_{T_2}\,.$$

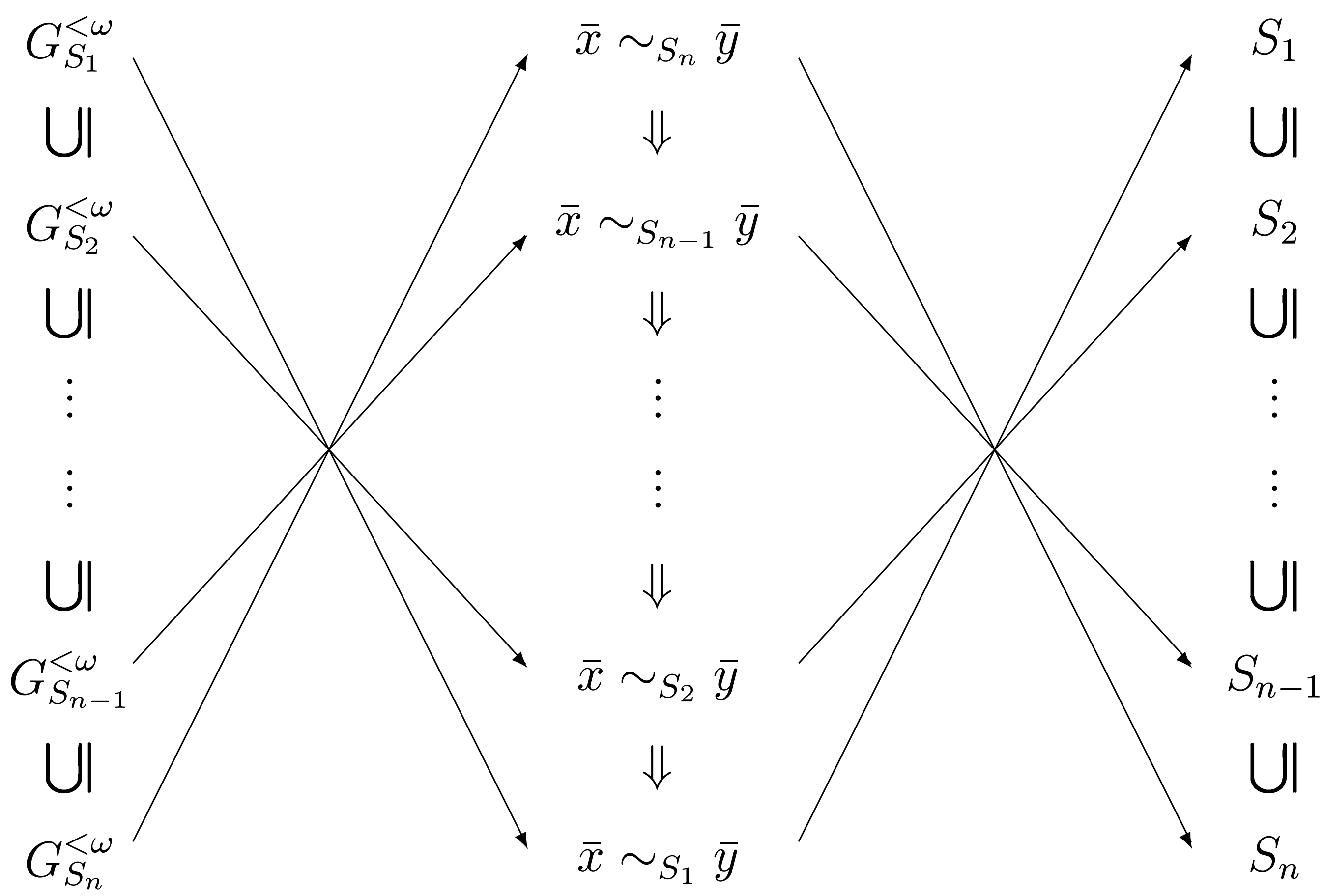
**Proposition 41.** *Let* $\Gamma$ *be a countable prs model of Presburger arithmetic and suppose that* $\mathrm{Aut}(\Gamma)$ *has trivial centre. Then* $\mathrm{Aut}(\Gamma)$ *has* $2^{\aleph_0}$ *closed normal subgroups.*

**Definition 42.** *Let* $\Gamma_{\mathsf{v}} \subseteq \bigcup_{n\in\omega} \Gamma^n$ *be the set of tuples* $\bar{x}$ *with* $\mathsf{v}(x_1) < \cdots < \mathsf{v}(x_n)$*.*

**Definition 43.** *If* $T$ *is* stQ *closed and* $\bar{a}, \bar{b} \in \widetilde{\Gamma_{\mathsf{v}}}$ *we say that* $\bar{a} \sim_T \bar{b}$ *if* $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ *and*

$$\mathrm{st} \quad \frac{a_1}{b_1}\Bigg), \ldots, \mathrm{st}\left(\frac{a_n}{b_n}\right)\Bigg) \in T.$$

**Lemma 44.** *Suppose* $\bar{a}, \bar{b} \in \Gamma_{\mathsf{v}}$*. Then* $\bar{a} \sim_T \bar{b}$ *if and only if* $\bar{a}g = \bar{b}$ *for some* $g \in G_T$*.*


$G^{<\omega}_{S_1}$
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$\bar{x} \sim_{S_n} \bar{y}$
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$S_{n-1}$
$\cup|$
$S_n$

**Theorem 45.** *In the diagram:*

1. $G_{T_1} \subseteq G_{T_2}$ *if and only if for all* $\overline{a}, \overline{b} \in \Gamma_\vee$ *we have* $\overline{a} \sim_{T_1} \overline{b} \quad \Rightarrow \overline{a} \sim_{T_2} \overline{b}$.

2. *The arrows are bijections.*

**Proposition 46.** *Suppose $\sim$ is a $G$invariant equivalence relation on $\Gamma_{\mathsf{v}}$ and that*

1. *if $a_1, \ldots, a_n \sim b_1, \ldots, b_n$ and $m \leq n$ then*

$$a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_n \sim b_1, \ldots, b_{m-1}, b_{m+1}, \ldots, b_n;$$

2. *if $a_1, \ldots, a_n \sim b_1, \ldots, b_n$ and $m \leq n+1$ then there is at least one pair $a'_m, b'_m$ with*

$$a_1, \ldots, a_{m-1}, a'_m, a_m, \ldots, a_n \sim b_1, \ldots, b_{m-1}, b'_m, b_m, \ldots, b_n;$$

3. *suppose $\overline{a}$,$\overline{b}$ are such that $\overline{a} \frown \overline{b}$, then $\overline{a} \sim \overline{b}$.*

*Then there is an* stQ*closed $T$ with $\overline{a} \sim \overline{b}$ if and only if $\overline{a} \sim_T \overline{b}$.*