Presburger Arithmetic and Pseudo-Recursive Saturation

by

David Llewellyn-Jones

A thesis submitted to the University of Birmingham
for the degree of PhD in the Faculty of Sciences

School of Mathematics and Statistics,

The University of Birmingham,

Edgbaston, Birmingham, B15 2TT.

November 19th, 2001

Abstract

The first half of this thesis looks at well known general properties of Presburger arithmetic, including quantifier elimination, types, compactness and homogeneity. It is accessible to the algebraist as well as the model theorist.

Let Γ be a model of Presburger arithmetic. Define the residue map $\varrho\colon \Gamma \to \widehat{\mathbb{Z}}$ sending an element to the sequence of its remainders and the standard part $\mathrm{st}\left(\frac{\gamma_1}{\gamma_2}\right) \in \mathbb{R} \cup \{\pm\infty\}$ for $\gamma_1, \gamma_2 \in \Gamma$ to be the supremum of the set $\left\{\frac{r}{s} \in \mathbb{Q} : rb < sa\right\}$. Define a partitioning of our model by the equivalence relation $\gamma_1 \equiv \gamma_2$ if and only if $\mathrm{st}\left(\frac{\gamma_1}{\gamma_2}\right) \notin \{0, \pm\infty\}$ and let $\mathrm{v}\colon \Gamma \to \Gamma/\equiv$ be the natural valuation.

We say that Γ is pseudo-recursively saturated if: the inverse image of the residue map is dense in $\Gamma/\mathbb{Z}$; for $x, y, z \in \Gamma$ there exists $w \in \Gamma$ such that $\mathrm{st}\left(\frac{w}{z}\right) = \mathrm{st}\left(\frac{x}{y}\right)$; and $\Gamma/\equiv$ forms a dense linear order with least point 0 and no greatest point.

We prove that pseudo-recursive saturation implies homogeneity and give results in the opposite direction indicating an affinity between the two.

Our main result concerns the automorphism group, G, of the countable pseudo-recursively saturated models of Presburger arithmetic, giving a correlation between the closed normal subgroups of G and sets of tuples of the standard parts of the model. We define $S \subseteq \bigcup_{n\in\omega} \left\{\mathrm{st}\left(\frac{\gamma_1}{\gamma_2}\right) : \gamma_1, \gamma_2 \in \Gamma\right\}^n$ to be stQ-closed if: all subsets of S defined to be those tuples of a certain length form a group; if $(r_1, \ldots, r_n) \in S$ and $m \leq n$ then $(r_1, \ldots, r_{m-1}, r_{m+1}, \ldots, r_n) \in S$; and similarly if $m \leq n+1$ then there exists some r'_m such that $(r_1, \ldots, r_{m-1}, r'_m, r_m, \ldots, r_n) \in S$. We then have that N is a closed normal subgroup of G if and only if N preserves some stQ-closed set S or $N = G$.

From this we are able to provide some results about the closed normal subgroups of G and to present a pair of Galois connections between closed normal subgroups of G, stQ-closed subsets of the set of standard parts and equivalence relations on Γ.

Acknowledgements

First and foremost I would like to thank my supervisor, Dr. Richard W. Kaye, for his continual help and guidance, and for introducing me to the engaging topic of Presburger arithmetic.

I would also like to thank Professor John Wilson, my internal examiner, and Dr. Dugald Macpherson, my external examiner, both for their helpful comments regarding my MPhil. Qual thesis. John Graham and Martin Quick also provided some very useful discussion with regard to proposition 14.1.6.

I am greatly indebted to Mr. Donn Dymock of the Deanes School whose enthusiasm for mathematics and dedication to teaching were nothing short of inspirational for me.

My lovely wife, Joanna Cargill, deserves especial thanks for supporting me in every possible way, and for taking an interest even when I wasn't being very interesting.

Finally but not least, I thank the School of Mathematics and Statistics for financial assistance.

Contents

Chapter 1

Introduction

1.1 Preamble

In this thesis we concern ourselves with the theory of Presburger arithmetic. This theory is best described in the same manner Mojżesz Presburger first described it prior to it acquiring his name, *viz.* as a system of arithmetic of whole numbers in which addition occurs as the only operation. The precise axiomatization which we will be using is given immediately at the outset of chapter 2, and reference to this will reveal that our arithmetic is also to have a discrete linear order imposed on it. We take the axiomatization as our starting point since we will be primarily concerned with non-standard models. We therefore wish to avoid any unnecessary reference to what we might regard as our *preconceived* ideas about the nature of arithmetic with addition. It is perhaps more usual, as in Smoryński [52], to approach the problem from the opposite direction, deriving the axioms after having achieved quantifier elimination. Nonetheless we do not reach this point until chapter 4. The method used to tackle the quantifier elimination here, and taken from work done by Richard Kaye [31], is innovative in that it is primarily algebraic in nature, rather than being excessively logic based. In light of this there are few prerequisites to the understanding of this thesis besides an awareness of basic algebraic and logical techniques. The only exceptions are perhaps in section 3.2, where for the sake of simplicity we use an elementary result from model theory, and in chapter 9 during a brief discussion of recursive saturation. For the most part, then, the background material found in chapter 0 of Kaye [29] will suffice.

The bulk of this thesis, then, can be seen as an analysis of Presburger arithmetic from an algebraic perspective. With this in mind, the first part – up to chapter 7 –

re-examines well known results regarding Presburger arithmetic in a new way. Again, much of the work in these chapters can be attributed to Richard Kaye [31]. In particular, the last part of chapter 6 contains an interesting result of Kaye's which shows that $\widehat{\mathbb{Z}}$ can be considered as a Presburger group if provided with a suitable ordering. Thus $\widehat{\mathbb{Z}}$ provides us with a useful example of a nonstandard model of Presburger arithmetic (we use nonstandard here in the model-theoretic sense, in this context meaning any model which is not $\mathbb{Z}$).

From chapter 9 onwards we begin to concentrate on the automorphisms of models of Presburger arithmetic, and in particular countable models which satisfy the requirement of being pseudo-recursively saturated. Pseudo-recursive saturation is closely linked to the notion of homogeneity, and we take some time in chapter 9 to discuss how the two are connected and why the properties of pseudo-recursive saturation are ideally suited for an analysis of the automorphism group. The intention has been to examine the automorphism group, G, of such models by considering the closed normal subgroups of G. Thus we look at the conjugates of automorphisms in chapter 10. Here we discover that we can achieve surprisingly much by taking conjugates of an automorphism. For value-defying automorphisms (those which permute elements by large distances) the situation is straightforward. For value-preserving automorphisms (which permute elements whilst preserving a particular partitioning of the models, as described in chapter 8) the situation is more complicated, but nonetheless is made far simpler by the result (theorem 11.5.4) that by conjugating elements we can map an element to any position which is 'close' to its original position. What it means for an element to be close is given a strict definition in the text, but effectively means that the new position cannot be distinguished from the original element by considering real multiples of it. It becomes clear from this that what is important when examining closed normal subgroups is precisely the set of standard parts of a model, or the ratios between elements. All of this is looked at carefully in chapters 10 through 12, and we are able to give a complete correlation between the closed normal subgroups and particular (defined) sets of standard parts from the model.

Using this correlation we conclude with various consequences relating equivalence relations, the closed normal subgroups and the sets of standard parts, along with some other results about the closed normal subgroups. It is hoped that these may be useful in discovering more about the automorphism group and hence the original Presburger groups themselves, in particular those models which are homogeneous or

Figure 1.1: Mojżesz Presburger.

pseudo-recursively saturated.

1.2 Literature survey

Mojżesz Presburger published his Masters Thesis, entitled 'On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation' in 1929. The importance of this paper — as the title suggests — was in proving the completeness of what is now known as Presburger arithmetic, the structure of which is considered in this thesis. An English translation of Presburger's original paper has been published by Jacquette [46]. The historical development of Presburger's work is chronicled by Zygmunt [58]. Other historical articles which include the work of Presburger in other contexts include that by Girant [21] who mentions Presburger in relation to his supervisor Alfred Tarski, Murawski [43] in an assessment of the contribution made by Polish mathematicians, which include Presburger, to decidability theory, and Woleński [57] who refers to Presburger in relation to the Warsaw school of logic. Figure 1.1 shows a photograph of a relatively young Presburger.

Although Presburger's result is significant in itself, much of the interest in Presburger arithmetic has grown through further results in complexity theory and its use in automated theorem proving. Major work was done on this in the 1970s, with papers by Cooper [14], Fischer and Rabin [18], and Oppen [45]; the last of the three proving an upper bound to the complexity of Presburger arithmetic of $2^{2^{2^{pn}}}$. A number of optimisations have allowed applied techniques to be established such as the Coq proof assistant [6] and those described by Rincón and Gadelha [47]. In general these optimisations consider restrictions on the form of sentences such as that by Schöning [50] who considers for example formulas with quantifier prefix of the form $\exists_1 \forall_2 \ldots \exists_m \forall^3$, Grädel [23] who proves that the class of $\exists\forall^*$-formulas has exponential complexity, or Morieka, Shibata, Naoki, Higashino and Taniguchi [42] who present a decision procedure for Σ_1-prenex normal form sentences. However work has also been done which considers the complexity and decidability of extensions of Presburger arithmetic. In general the aim is to find decidable extensions of the language which have minimal effect on the complexity of the theory. Korec [34] presents a decidable extension of Presburger arithmetic $(\mathbb{N}, B_2, +)$ where B_2 represents the Pascal triangle modulo 2, and Artëmov and Montagna [2] consider an extension with provability operator as an example to a wider set of results. Smoryński [52] and Hosono [28] have independently demonstrated the decidability of SA, an extension of Presburger arithmetic to the rational numbers. Other papers on extensions of Presburger arithmetic include Halpern [26], Cantone, Cutello and Schwartz [7] and Weispfenning [56].

A result due to Buchi [9] that Presburger arithmetic can be simulated by finite automata over infinite words has led to a number of results from Michaux and Villemaire working together on Cobham's theorem. This theorem states that if k and l are multiplicatively independent and $S \subseteq \mathbb{N}$ is both k and l-recognisable, then S is recognisable. Their 1996 paper [40] contains a survey of the problem and a number of further papers include results by them regarding Cobham's theorem and Semenov's extension of it [39, 41].

Many of the papers listed above serve to justify the study of Presburger arithmetic in a more practical sense. However Presburger arithmetic can also be seen as an extension of alternative structures about which a large stock of interesting results exist. For example the intimate relationship between Presburger arithmetic and linearly ordered divisible abelian groups (cf. Chapter 3) is reaffirmed by Robinson [48] who has proven that a theory of linearly ordered abelian groups has quantifier elimination if and only if it is the theory of linearly ordered divisible abelian groups. It is this property of Presburger arithmetic being an ordered abelian group that allows us to give a version of the Hahn embedding theorem in the appendix (section 14.1). This asserts that an ordered abelian group can be embedded in a lexicographic product of an appropriate number of copies of $\mathbb{R}$, proofs of which can be found in Fuchs [20] and Fleischer [19] both of which include more general information about the theorem. Glass [22] contains a proof which reflects the case of Presburger arithmetic well, although it is still concerned with the more general case of ordered abelian groups. It also contains references to other relevant material, including the original paper by Hahn [25]. Many specific generalisations of the Hahn embedding theorem exist, such as those by Conrad [13] who proves the theorem for abelian lattice-ordered groups, Asmar and Montgomery-Smith [3] who show the result for locally compact groups G where $\widehat{G}$ contains a measurable order, or Gravett [24] who proves the result for valued linear spaces which is of particular interest.

Apart from linearly ordered divisible abelian groups, Presburger arithmetic shares properties with the theory of dense linear orderings with colourings and the theory of valued fields. Well known results arising from work by Ax and Kochen [4, 5], and Ershov [17] tell us that the model theory of a valued field is dependent on the model theory of its value group and residue class rings. As a complete decidable theory, Presburger arithmetic therefore constitutes an appealing value group, not least since $\mathbb{Z}$ is often a natural choice. Many results concerning the model theory of valued fields can be found in Weispfenning [55], a paper which would appear to cover results from a number of other sources such as van den Dries [53] or earlier work by Weispfenning [54]. The fact that the quotient of a Presburger group with its embedded copy of $\mathbb{Z}$ is divisible makes the results of the aforementioned paper especially applicable. There also exists a paper by Delon [15] claiming to provide a less technical account of the subject than those cited above.

A related area in which Presburger results may have a bearing is that of Λ-trees. If Λ is an ordered abelian group a Λ-tree constitutes a tree-like Λ-metric space which satisfies certain closure properties. Roger Alperin and Hyman Bass's paper [1] provides a good introduction to the topic of Λ-trees. Their relevance to Presburger arithmetic lies in the fact that as an ordered abelian group, Presburger arithmetic is a potential distance map codomain, and as such also forms a (linear) Λ-tree itself. A significant number of papers relating to Λ-trees have been published by Ian Chiswell including several [10, 11, 12] which consider results concerning group actions on Λ-trees.

Finally the analogies between Presburger arithmetic and Peano arithmetic cannot be ignored, not least because of the large body of relevant work relating to Peano arithmetic which exists. Recursively saturated models in particular from the basis for the introduction of pseudo-recursive saturation in this thesis, and the relationship between these notions is discussed in Chapter 9. In a paper by Richard Kaye [32] the normal subgroups of the automorphism group of countable recursively saturated models of Peano arithmetic are classified, a result which he has subsequently extended further [30]. This work constituted the impetus for much of the work found here, although the model theoretic structure of Presburger arithmetic has solicited largely unrelated methods to those used for Peano arithmetic. The collection of which the previously cited paper forms a part also contains a significant amount of material relevant to automorphism groups more generally and for more information about countable recursively saturated models of Peano arithmetic see Smoryński [51], Kotlarski [38, 37] and Kossak and Kotlarski [35]. Kossak and Schmerl [36] write about the automorphisms of arithmetically saturated models of Peano arithmetic, which form a subclass of the recursively saturated models. For a much wider scope of topics relating to the model theory of Peano arithmetic including (amongst other things) more introductory material see Kaye [29].

Chapter 2

The Properties of a Presburger Group

2.1 The Presburger axioms

A model of Presburger Arithmetic has the form $(\Gamma, +, <, 0, 1)$, where Γ is an abelian group with the usual group axioms, and $+$ is the binary operation on the group. The order $<$ is linear and respected by $+$, so along with the standard axioms for an abelian group, a model of Presburger Arithmetic (a *Presburger Group*) will also satisfy the following:

$\mathbf{A_1}$: $\forall x, y, z(x < z \rightarrow x + y < y + z)$; (order respects $+$)

$\mathbf{A_2}$: $\exists x(x > 0 \wedge \forall y(y > 0 \rightarrow y \geq x))$; (discrete ordering)

We distinguish the unique element x satisfying $\mathbf{A_2}$ by assigning it the constant term 1.

Notation. If $n \in \mathbb{Z}$ and $x \in \Gamma$ we denote nx to be the following:

$$nx = \begin{cases} x + x + \cdots + x \quad (n \text{ times}) & \text{when } n > 0, \\ 0 \quad (\text{the group identity}) & \text{when } n = 0, \\ (-x) + (-x) + \cdots + (-x) \quad (-n \text{ times}) & \text{when } n < 0. \end{cases}$$

Using this notation we may write our final axiom schema — that all congruence classes of mod n be defined — as follows:

$\mathbf{A_3}$: $\forall x \in \Gamma \, \exists y \in \Gamma \, \exists i \in \{0, 1, \ldots, n-1\}(x = ny + i)$ for each $n \in \mathbb{N}, n \geq 1$.

2.2 Some basic properties of Presburger groups

1. Using the group axioms and discrete ordering, we obtain the following:

   (a) $x > y \quad \Rightarrow \quad x \geq y + 1$;

   (b) $nx = 0 \quad \Rightarrow \quad n = 0$ or $x = 0$;

   (c) $x = ny > 0 \quad \Rightarrow \quad x \geq n$ and $x \geq y$.

2. Using the Presburger axioms we obtain

   (a) The congruence class of $x \bmod n$ is unique;

   (b) The quotient x/n is unique (when it exists).

Definition 2.2.1. Set $\widehat{\mathbb{Z}}$ to be the group defined on the set of sequences $(r_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned} 0 \leq r_n < n \qquad & \text{for all } n \geq 1, \\ r_{nm} \equiv r_n \pmod{n} \quad & \text{for all } n, m \geq 1. \end{aligned}$$

If $(r_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ are of this form then the group action $+$ is defined as follows:

$$(r_n)_{n\in\mathbb{N}} + (s_n)_{n\in\mathbb{N}} = (r_n + s_n \pmod{n})_{n\in\mathbb{N}}.$$

The zero element is then $(0, 0, \dots)$ and for an element $(r_n)_{n\in\mathbb{N}}$ the inverse element $(r_n)^{-1}_{n\in\mathbb{N}}$ is

$$(-r_n \pmod{n})_{n\in\mathbb{N}} = (n - r_n)_{n\in\mathbb{N}}.$$

We can also define $\widehat{\mathbb{Z}}$ by taking the set $(\mathbb{Z}_n)_{n\in\mathbb{N}}$ partially ordered with respect to the relation

$$\mathbb{Z}_n \leq \mathbb{Z}_m \quad \Longleftrightarrow \quad n \text{ divides } m.$$

If we also take $f_{nm} \colon \mathbb{Z}_m \to \mathbb{Z}_n$ to be the map defined by

$$f_{nm}(x) = \text{ the unique } i \in \mathbb{Z}_n \text{ such that } ny + i = x \text{ for some } y \in \mathbb{Z}_m,$$

then we can define $\widehat{\mathbb{Z}}$ to be the inverse limit of the family $(\mathbb{Z}_n)_{n\in\mathbb{N}}$ with respect to the family of mappings (f_{nm}). We write $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}_n$.

By applying the notation given in section 2.1 to the distinguished element 1 we see that there is a canonical embedding of the integers $\mathbb{Z}$ inside every Presburger model.

More specifically, if Γ is a Presburger group we intend to identify $\mathbb{Z}$ with its image under the map $\mathbb{Z} \to \Gamma$ defined by

$$n \mapsto n.1$$

where $n \in \mathbb{Z}$ and $1 \in \Gamma$ have already been defined.

The preceding facts ensure that the following is well defined

Definition 2.2.2. Suppose Γ is a Presburger group. Then the **natural residue map**

$$\varrho\colon \Gamma \to \widehat{\mathbb{Z}}$$

is defined to be the homomorphism such that

$$\varrho(\gamma) = (\gamma \pmod 1, \gamma \pmod 2, \gamma \pmod 3, \dots)$$

for all $\gamma \in \Gamma$. The image of an element $\gamma \in \Gamma$ under ϱ will be referred to as the **residue** of γ.

We note that for this map $\varrho(1) = \widehat{1}$ where $1 \in \mathbb{Z} \subseteq \Gamma$ and $\widehat{1} \in \widehat{\mathbb{Z}}$. Consequently $\varrho(n \cdot 1) = n \cdot \widehat{1}$ and we can therefore use the same method as that for Γ to embed a copy of $\mathbb{Z}$ inside $\widehat{\mathbb{Z}}$. Again, we identify $\mathbb{Z}$ with the image of this embedding.

Definition 2.2.3. Let $r \in \widehat{\mathbb{Z}}$ be the sequence $(r_n)_{n\in\mathbb{N}}$. We define $\Gamma[r]$ to be the set of all expressions

$$\lambda + \frac{(X - r_n)}{n}\mu \quad \text{where } \lambda \in \Gamma, \mu \in \mathbb{Z} \text{ and } n \in \mathbb{N} \text{ with } n \geq 1$$

and so that

$$\lambda + \frac{(X - r_n)}{n}\mu = \lambda' + \frac{(X - r_m)}{m}\mu'$$
$$\overset{\text{def}}{\iff} \quad m\mu = n\mu' \text{ and } nm\lambda' - nr_m\mu' = nm\lambda - mr_n\mu \text{ in } \Gamma.$$

We note that n divides $r_{nm} - r_n$ since $r_{nm} \equiv r_n \pmod n$. Hence we see that

$$\frac{(X - r_n)}{n}\mu = \frac{(X - r_{nm})}{nm}m\mu + \frac{(r_{nm} - r_n)}{n}\mu,$$

and so we may define the operation of addition to be as follows:

$$\left(\lambda + \frac{(X - r_n)}{n}\mu\right) + \left(\lambda' + \frac{(X - r_m)}{m}\mu'\right)$$
$$= \left(\lambda + \lambda' + \frac{(r_{nm} - r_n)}{n}\mu + \frac{(r_{nm} - r_m)}{m}\mu'\right) + \frac{(X - r_{nm})}{nm}(m\mu + n\mu').$$

We hope to show that $\Gamma[r]$ is a Presburger group. In order to do this, we must show that it satisfies the Presburger axioms as given earlier.

We first show that $\Gamma[r]$ is an abelian group. Addition is clearly commutative, since

$$\begin{aligned}&\left(\lambda + \frac{(X - r_n)}{n}\mu\right) + \left(\lambda' + \frac{(X - r_m)}{m}\mu'\right)\\ &\quad = \left(\lambda + \lambda' + \frac{(r_{nm} - r_n)}{n}\mu + \frac{(r_{nm} - r_m)}{m}\mu'\right) + \frac{(X - r_{nm})}{nm}(m\mu + n\mu'),\end{aligned}$$

so by commutativity of addition in Γ and multiplication in $\mathbb{Z}$, we have

$$\begin{aligned}&= \left(\lambda' + \lambda + \frac{(r_{mn} - r_m)}{m}\mu' + \frac{(r_{mn} - r_n)}{n}\mu\right) + \frac{(X - r_{mn})}{mn}(n\mu' + m\mu).\\ &= \left(\lambda' + \frac{(X - r_m)}{m}\mu'\right) + \left(\lambda + \frac{(X - r_n)}{n}\mu\right)\end{aligned}$$

as required. The identity element $0 \in \Gamma[r]$ is given when $\lambda = 0$ (in Γ), $\mu = 0$ (in $\mathbb{Z}$) and n is an arbitrary element of $\mathbb{N}$. It is easy to see that this is well defined. We then have

$$\begin{aligned}&\left(0 + \frac{(X - r_n)}{n}0\right) + \left(\lambda' + \frac{(X - r_m)}{m}\mu'\right)\\ &\quad = \left(0 + \lambda' + \frac{(r_{nm} - r_n)}{n}0 + \frac{(r_{nm} - r_m)}{m}\mu'\right) + \frac{(X - r_{nm})}{nm}(m.0 + n\mu')\\ &\quad = \left(\lambda' + \frac{(r_{nm} - r_m)}{m}\mu'\right) + \frac{(X - r_{nm})}{m}\mu'\\ &\quad = \left(\lambda' + \frac{(X - r_m)}{m}\mu'\right),\end{aligned}$$

and we see that this is indeed the identity element. For an element

$$\left(\lambda + \frac{(X - r_n)}{n}\mu\right)$$

we take its inverse element to be

$$\left(-\lambda + \frac{(X - r_n)}{n}(-\mu)\right)$$

where $-\lambda$ constitutes the inverse element of λ in Γ. Adding these two elements together, we get:

$$\begin{aligned}
&\left(\lambda + \frac{(X - r_n)}{n}\mu\right) + \left(-\lambda + \frac{(X - r_n)}{n}(-\mu)\right) \\
&\quad = \left(\lambda - \lambda' + \frac{(r_{n^2} - r_n)}{n}\mu + \frac{(r_{n^2} - r_m)}{m}(-\mu)\right) + \frac{(X - r_{n^2})}{n^2}(n\mu - n\mu) \\
&\quad = \left(0 + \frac{(r_{n^2} - r_n)}{n}(\mu - \mu)\right) + 0 \\
&\quad = 0 + \frac{(r_{n^2} - r_n)}{n}0,
\end{aligned}$$

which by our previous definition, is the identity element. By noting finally that associativity follows directly from associativity in Γ we see that $\Gamma[r]$ is indeed an abelian group.

We may apply a discrete ordering to $\Gamma[r]$ by defining

$$\lambda + \frac{(X - r_n)}{n}\mu > 0 \quad \Longleftrightarrow \quad \mu > 0 \text{ or } (\mu = 0 \text{ and } \lambda > 0).$$

With this ordering we note that $1 + \frac{(X-r_n)}{n}.0 = 1$ is the least positive element.

In order to show that $\Gamma[r]$ is a Presburger group, we must finally show that for every element $\gamma \in \Gamma[r]$, and every number $n \in \mathbb{N}$, the element γ has a residue $\bmod\, n$. In order to do this we note that X is an element of $\Gamma[r]$, since $X = r_n + \frac{(X-r_n)}{n}.n$, and that therefore $X \equiv r_n \pmod{n}$. Now, take an arbitrary element $\gamma = \lambda + \frac{(X-r_n)}{n}\mu$, and consider the $\frac{(X-r_n)}{n}\mu$ part. We want to show that for $m \in \mathbb{N}$ this has some residue $\bmod\, m$. Now

$$\begin{aligned}
\frac{(X - r_n)}{n}\mu &= \frac{(X - r_{nm})}{nm}m\mu + \frac{(r_{nm} - r_n)}{n}\mu, \\
&= m.\frac{(X - r_{nm})}{nm}\mu + \frac{(r_{nm} - r_n)}{n}(m\nu + s) \qquad (2.1) \\
&\qquad \text{where} \quad \mu = m\nu + s, 0 \leq s < m.
\end{aligned}$$

The latter half of equation (2.1) is well defined since $r_{nm} \equiv r_n \pmod{n}$. From this we may conclude that

$$\frac{(X - r_n)}{n}\mu \quad \equiv \quad \frac{(r_{nm} - r_n)}{n}s \pmod{m}.$$

Given that r_{nm}, r_n, n and s are all natural numbers, it is clear that $\frac{(X-r_n)}{n}\mu$ has a well defined residue $\bmod\, m$, and that therefore so does γ.

We may conclude from the above that $\Gamma[r]$ does indeed form a Presburger group. Furthermore, we observe that by identifying $\lambda \in \Gamma$ with the element $\lambda + \frac{(X-r_n)}{n}.0 \in \Gamma[r]$ we may consider Γ to be a convex subgroup of $\Gamma[r]$. For suppose we take an arbitrary element $\gamma' = \lambda + \frac{(X-r_n)}{n}\mu$ of $\Gamma[r]$. Clearly $\gamma' \in \Gamma$ if and only if $\mu = 0$. Hence if $\gamma' \notin \Gamma$ then either $\mu < 0$ or $\mu > 0$. In the case that $\mu < 0$ we see that $\gamma' < \gamma$ for all $\gamma \in \Gamma$. Similarly if $\mu > 0$ then $\gamma' > \gamma$ for all $\gamma \in \Gamma$. Hence Γ is clearly a convex subgroup of $\Gamma[r]$.

This process, whereby a Presburger group Γ is extended by an element $r \in \widehat{\mathbb{Z}}$, can be iterated any number of times.

Definition 2.2.4. If Γ is a Presburger group, we define the extension $\Gamma[r_1, r_2, \ldots, r_n]$ to be the iterated extension $\Gamma[r_1][r_2]\ldots[r_n]$ where each iteration is an extension as defined above.

Notation. In the construction of $\Gamma[r]$ given above in definition 2.2.3 we introduced an implicitly defined element X which was found to have residue r. We call this element the **extension element** of $\Gamma[r]$ and say that Γ has been **extended upwards** by X. The important aspect of this construction is that we are able to create models containing elements with any given residues and hence we may occasionally write $\Gamma[x]$ to mean Γ extended upwards with an element x having residue r, where this x is the extension element of the model.

Proposition 2.2.5. Γ is a Presburger group if and only if it is a discretely ordered abelian group and there is a group homomorphism:

$$\theta\colon \Gamma \to \widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}_n$$

(with induced maps $\theta_n\colon \Gamma \to \mathbb{Z}_n$) such that for all $x \in \Gamma$ and all $n \in \mathbb{N}$, $n > 1$,

$$\exists\, y \quad x = ny \qquad \Longleftrightarrow \qquad \theta_n(x) = 0.$$

(We may also add that $\theta\colon 1 \mapsto \widehat{1} \in \widehat{\mathbb{Z}}$).

Proof. ($\Longrightarrow$) Define $\theta_n\colon \Gamma \to \mathbb{Z}_n$ by

$$\theta_n(x) = \text{ the unique } i \in \{0, \ldots, n-1\} \text{ such that } ny + i = x \text{ for some } y \in \Gamma.$$

Clearly this is a homomorphism respecting the maps $f_{mn}\colon \mathbb{Z}_n \to \mathbb{Z}_m$ as defined earlier (where $m \mid n$), and by virtue of the fact that $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}_n$ it therefore induces a unique

$\theta\colon \Gamma \to \widehat{\mathbb{Z}}$. This θ will clearly satisfy the requirements as set out in the statement of the proposition. We are obviously relying here on the universal property of an inverse system of sets which can be found as proposition 1, III.7.2 of Bourbaki [8].

($\Longleftarrow$) Given θ as above, $x \in \Gamma$ and $n > 1$, we suppose that

$$\theta_n(x) \equiv i \pmod{n}.$$

Then

$$\theta_n(x + k) \equiv i + k\theta_n(1) \pmod{n}.$$

But $(\theta_n(1), n) = 1$, since otherwise we would have

$$n \mid r\theta_n(1) \quad \text{for some} \quad r < n$$

so that $\theta_n(r) = 0$, implying that n divides r in Γ, giving $0 < \frac{r}{n} < 1$.

Hence $k\theta_n(1) \equiv -i$ for some $k < n$ so $\theta_n(x + k) \equiv 0$. But this gives us that $ny = x + k$ for some $y \in \Gamma$ and hence $n(y - 1) + (n - k) = x$ as required. □

Chapter 3

Divisible Ordered Abelian Groups

3.1 From Presburger groups to divisible ordered abelian groups

Definition 3.1.1. An abelian group Γ is **divisible** if for all $x \in \Gamma$ and all $n > 0$ from $\mathbb{N}$ there is $y \in \Gamma$ with $ny = x$.

Example. $\Gamma = (\mathbb{Q}, +)$

This definition applies to any group, but since a Presburger group is by definition ordered abelian, we will be considering the ordered abelian groups in particular.

An ordered abelian divisible group cannot have a least positive element (*i.e.* cannot be discrete) since

$$n > 1, \quad x > 0, \quad ny = x \quad \Rightarrow \quad 0 < y < x,$$

for $y \geq x \geq 0$ implies $ny \geq x$ and $y \leq 0$ implies $ny \leq 0$.

In the previous chapter we noted that any Presburger group Γ has a copy of $\mathbb{Z}$ embedded in it. By the discreteness of the ordering on Γ this copy of $\mathbb{Z}$ is convex in Γ and it can be seen immediately that it is also a normal subgroup. We can therefore take the quotient $\Gamma/\mathbb{Z}$ to obtain an ordered group with addition defined in the obvious manner:

$$x/\mathbb{Z} + y/\mathbb{Z} \stackrel{\text{def}}{=} (x+y)/\mathbb{Z}.$$

This is clearly well defined. It is straightforward to check that $\Gamma/\mathbb{Z}$ is an ordered abelian group which is divisible from the axiom schema $\mathbf{A_3}$. For given any $\mathbb{Z} + x \in \Gamma/\mathbb{Z}$ and

$n \in \mathbb{N}$ this axiom tells us that there is some $\mathbb{Z} + y \in \Gamma/\mathbb{Z}$ and $i \in \mathbb{N}$ such that $\mathbb{Z} + x = n(\mathbb{Z} + y) + i = n(\mathbb{Z} + y)$. We formalize this in the following proposition:

Proposition 3.1.2. Any Presburger group Γ has a group homomorphism

$$\theta \colon \Gamma \to D$$

to some ordered divisible abelian group D, which respects $\leq$, in the sense that

$$x \leq y \text{ in } \Gamma \quad \Rightarrow \quad \theta(x) \leq \theta(y) \text{ in } D,$$

and which has kernel

$$\ker \theta \cong (\mathbb{Z}, +, <).$$

Proof. Follows from the previous remarks. □

We observe that $\Gamma/\mathbb{Z}$ has an induced residue map

$$\varrho/\mathbb{Z} \colon \Gamma/\mathbb{Z} \to \widehat{\mathbb{Z}}/\mathbb{Z}.$$

Moreover, since $\Gamma/\mathbb{Z}$ is divisible and torsion free (for all $n \in \mathbb{Z}$ and $\gamma \in \Gamma$ if $n\gamma = 0$ then $n = 0$ or $\gamma = 0$), it is a vector space over $\mathbb{Q}$.

3.2 From divisible ordered abelian groups to Presburger groups

The previous section showed that we can reduce every Presburger group to a some divisible ordered abelian group by factoring by $\mathbb{Z}$. The next theorem shows the reverse, that given certain minimal conditions applying to a divisible ordered abelian group, we can find a Presburger group which has this divisible ordered abelian group as its quotient after factoring by $\mathbb{Z}$. Although the proof appears relatively long, it in fact just involves a large quantity of straightforward case checking in order to apply Zorn's lemma.

Theorem 3.2.1. Suppose we have a divisible ordered abelian group $\widetilde{\Gamma}$ and a homomorphism $\widetilde{\varrho} \colon \widetilde{\Gamma} \to \widehat{\mathbb{Z}}/\mathbb{Z}$. Then there exists a Presburger group Γ and an isomorphism $\theta \colon \Gamma/\mathbb{Z} \to \widetilde{\Gamma}$ so that the diagram

$$
\begin{array}{ccccc}
\Gamma & \longrightarrow & \Gamma/\mathbb{Z} & \xrightarrow{\theta} & \widetilde{\Gamma} \\
\big\downarrow \varrho & & & & \big\downarrow \widetilde{\varrho} \\
\widehat{\mathbb{Z}} & & \longrightarrow & & \widehat{\mathbb{Z}}/\mathbb{Z}
\end{array}
\qquad \text{commutes}
$$

where $\varrho\colon \Gamma \to \widehat{\mathbb{Z}}$ is the residue map of Γ and the unmarked arrows represent natural quotient maps.

Proof. We prove this using Zorn's lemma applied to the poset

$$
\mathscr{P} = \left\{ (X,\theta) : \begin{array}{l}
X \text{ is a Presburger group, } \operatorname{dom}(X) \subseteq V_{\operatorname{card}|\widetilde{\Gamma}|} \\[2ex]
X \xrightarrow{\eta} X/\mathbb{Z} \overset{\theta}{\cong} \widetilde{X} \leq \widetilde{\Gamma} \\[2ex]
\text{and} \quad \begin{array}{ccccc}
X & \longrightarrow & X/\mathbb{Z} & \xrightarrow{\theta} & \widetilde{X} \\
\big\downarrow \varrho & & & & \big\downarrow \widetilde{\varrho} \\
\widehat{\mathbb{Z}} & & \longrightarrow & & \widehat{\mathbb{Z}}/\mathbb{Z}
\end{array} \quad \text{commutes}
\end{array} \right\},
$$

with the partial order defined by inclusion as follows:

$$
(X,\theta) \leq (X',\theta') \iff X \leq X' \quad \text{and} \quad \theta = \theta' \restriction_{X/\mathbb{Z}} . \tag{3.1}
$$

Note that although the class of all Presburger groups is too large to be a set, the restriction of X to those Presburger groups with domain contained in V_λ, where $\lambda = \operatorname{card}|\widetilde{\Gamma}|$, ensures that $\mathscr{P}$ nonetheless is. Since we are only interested in Presburger groups up to isomorphism and with $\operatorname{card}|X| \leq \operatorname{card}|\widetilde{\Gamma}|$, the set V_λ will certainly contain every Presburger group required up to isomorphism for the purposes of this theorem.

In order to apply Zorn's lemma we must first show that every chain in $\mathscr{P}$ has an upper bound. So suppose that $(X_i,\theta_i)_{i\in I}$ is a chain in $\mathscr{P}$. We want to show that if $X = \bigcup_{i\in I} X_i$ and $\theta = \bigcup_{i\in I} \theta_i$ (where this θ is well defined by equation (3.1)) then $(X,\theta) \in \mathscr{P}$. Now by an elementary result of model theory, if $\mathscr{U}_\epsilon$, $\epsilon < \alpha$ is a set of models with $\mathscr{U}_\epsilon \models T$ for all $\epsilon < \alpha$, then $\cup_{\epsilon<\alpha} \mathscr{U}_\epsilon \models T$ provided T is $\forall\exists$-axiomatizable (see for example [33], theorem 3.1.13). Reference to the Presburger axioms (section 2.1) reveals that they are indeed in $\forall\exists$ form, from which it immediately follows that X satisfies the Presburger axioms. It is also clear that

$$
X \xrightarrow{\eta} X/\mathbb{Z} \overset{\theta}{\cong} \widetilde{X} \leq \widetilde{\Gamma}
$$

is well defined and that $\widetilde{X}$ is a divisible subgroup of $\widetilde{\Gamma}$. We want to show that the diagram

$$\begin{array}{ccccc} X & \longrightarrow & X/\mathbb{Z} & \xrightarrow{\theta} & \widetilde{X} \\ \big\downarrow \varrho & & & & \big\downarrow \widetilde{\varrho} \\ \widehat{\mathbb{Z}} & & \longrightarrow & & \widehat{\mathbb{Z}}/\mathbb{Z} \end{array}$$

commutes. *I.e.* we must show that, for all $x \in X$,

$$\eta'(\varrho(x)) = \widetilde{\varrho}(\theta(\eta(x))),$$

where $\eta\colon \Gamma \to \Gamma/\mathbb{Z}$ and $\eta'\colon \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}/\mathbb{Z}$ are the natural quotient maps.

So let x be an arbitrary element from X. Then $x \in X_i$ for some $i \in I$, and $\eta'(\varrho_i(x)) = \widetilde{\varrho}_i(\theta_i(\eta(x)))$. But $\varrho_i = \varrho \restriction_{X_i}$ and $\theta_i = \theta \restriction_{X_i/\mathbb{Z}}$ so we clearly have that $\eta'(\varrho(x)) = \widetilde{\varrho}(\theta(\eta(x)))$ as required.

We have thus shown that every chain in $\mathscr{P}$ has an upper bound. We may therefore apply Zorn's lemma to get a maximal element (X, θ) for which we hope to show that

$$\theta\left(X/\mathbb{Z}\right) = \widetilde{\Gamma}.$$

So suppose otherwise. Then it is possible to find some $\widetilde{a} \in \widetilde{\Gamma}$ for which there is no $a \in X$ with $\theta(\mathbb{Z} + a) = \widetilde{a}$. We shall establish a contradiction by showing that this is not in fact possible. So we have that $\widetilde{\varrho}(\widetilde{a}) = r'$ for some $r' = \mathbb{Z} + r \in \widehat{\mathbb{Z}}/\mathbb{Z}$, where $r \in \widehat{\mathbb{Z}}$. Now $\widetilde{X}$ is a divisible group, so it is clear that $\widetilde{a} \notin \left\langle \widetilde{X} \right\rangle$ since $\left\langle \widetilde{X} \right\rangle = \widetilde{X}$.

Now we may extend our group X by r to get $X[a]$ where $\varrho(a) = r = (r_n)_{n \in \mathbb{N}}$ and with an isomorphism $\widehat{\theta}$ such that

$$X[a]/\mathbb{Z} \overset{\widehat{\theta}}{\cong} \left\langle \widetilde{X}, \widetilde{a} \right\rangle$$

in the following manner:-

Define $X[a]$ to be the set of expressions

$$\lambda + \frac{(x - r_n)}{n}\mu \quad \text{where } \lambda \in X, \mu \in \mathbb{Z}, \text{ and } n \in \mathbb{N} \text{ with } n \geq 1,$$

and so that

$$\begin{aligned} \lambda + \frac{(x - r_n)}{n}\mu = \lambda' + \frac{(x - r_m)}{m}\mu' \\ \overset{\text{def}}{\iff} \quad m\mu = n\mu' \text{ and } nm\lambda' - nr_m\mu' = nm\lambda - mr_n\mu \text{ in } X. \end{aligned}$$

This technique is the same as that described in definition 2.2.3, where a more comprehensive explanation is given. We take the operation of $+$ in $X[a]$ to be the same abelian binary operation as that in definition 2.2.3. It is however necessary for our construction to differ slightly when specifying the ordering. In this case we set

$$\lambda + \frac{(x - r_n)}{n}\mu > 0 \iff \mu \neq 0 \text{ and } \mu\widetilde{a} > \theta(\mathbb{Z} - n\lambda) \text{ in } \widetilde{\Gamma},$$
$$\text{or } \mu = 0 \text{ and } \lambda > 0 \text{ in } X.$$

This is well defined since if

$$\left(\lambda + \frac{(x - r_n)}{n}\mu\right) = \left(\lambda' + \frac{(x - r_m)}{m}\mu'\right)$$

and

$$\lambda + \frac{(x - r_n)}{n}\mu > 0$$

then by the definition either $\mu' \neq 0$ and $\mu\widetilde{a} > \theta(\mathbb{Z} - n\lambda)$ or $\mu = 0$ and $\lambda > 0$. We also know that $m\mu = n\mu'$ and $nm\lambda' - nr_m\mu' = nm\lambda - mr_n\mu$.

If $\mu' = 0$ then $\mu = 0$ so $\lambda = \lambda'$. If $\mu' \neq 0$ then

$$\begin{aligned} n\mu'\widetilde{a} = m\mu\widetilde{a} &> m\theta(\mathbb{Z} - n\lambda) \\ &= \theta(\mathbb{Z} - nm\lambda + mr_n\mu) \\ &= \theta(\mathbb{Z} - nm\lambda' - nr_m\mu') \\ &= \theta(\mathbb{Z} - nm\lambda'). \end{aligned}$$

In both cases it is clear that

$$\lambda' + \frac{(x - r_m)}{m}\mu' > 0$$

and hence the ordering is well defined.

We wish to construct a positive cone P such that for every $0 \neq \gamma \in X[a]$, either $\gamma \in P$ or $-\gamma \in P$ and with P closed under addition. So suppose $\gamma = \lambda + \frac{(x-r_n)}{n}\mu$ is some arbitrary element of $X[a]$. Clearly if $\mu = 0$ the order is defined entirely by the original ordering of $\lambda = \gamma$, and hence one of $\gamma > 0, \gamma = 0$ or $-\gamma > 0$ must hold. If $\mu \neq 0$ then since $a \neq 0$ we cannot have $\gamma = 0$. So suppose $\mu\widetilde{a} \not> \theta(\mathbb{Z} - n\lambda)$. Then $\mu\widetilde{a} \leq \theta(\mathbb{Z} - n\lambda)$ so $-\mu\widetilde{a} \geq \theta(\mathbb{Z} + n\lambda)$. But if $-\mu\widetilde{a} = \theta(\mathbb{Z} + n\lambda)$ then by divisibility of $X/\mathbb{Z}$ we have that $\widetilde{a} = -\frac{1}{\mu}\theta(\mathbb{Z} + n\lambda) \in \widetilde{X}$ which contradicts our assumptions. We therefore have $-\mu\widetilde{a} > \theta(\mathbb{Z} + n\lambda)$ from which $-\gamma = -\lambda + \frac{(x-r_n)}{n}(-\mu) > 0$ as required.

We also need to show that P is closed under addition. Consider

$$\gamma = \lambda + \frac{(x - r_n)}{n}\mu > 0, \qquad \gamma' = \lambda' + \frac{(x - r_m)}{m}\mu' > 0.$$

Then

$$\gamma + \gamma' = \left(\lambda + \lambda' + \frac{(r_{nm} - r_n)}{n}\mu + \frac{(r_{nm} - r_m)}{m}\mu'\right) + \frac{(x - r_{nm})}{nm}(m\mu + n\mu').$$

If $\mu = \mu' = 0$ then $\gamma + \gamma' = \lambda + \lambda' > 0$.

If $\mu \neq 0, \mu' = 0$ then $\gamma' = \lambda' > 0$ and $\mu\widetilde{a} > \theta(\mathbb{Z} - n\lambda)$. But then $m\mu + n\mu' \neq 0$ and $\mu\widetilde{a} > \theta(\mathbb{Z} - nm(\lambda + \lambda'))$ since $n, m, \lambda' > 0$. It follows that $\gamma + \gamma' > 0$ as required.

If $\mu = 0, \mu' \neq 0$ a symmetric argument holds.

If $\mu \neq 0, \mu' \neq 0$ then $\mu\widetilde{a} > \theta(\mathbb{Z} - n\lambda)$ and $\mu'\widetilde{a} > \theta(\mathbb{Z} - m\lambda')$. In this case if $m\mu + n\mu' \neq 0$, then

$$m\mu\widetilde{a} + n\mu'\widetilde{a} > \theta(\mathbb{Z} - nm\lambda) + \theta(\mathbb{Z} - nm\lambda')$$

so

$$(m\mu + n\mu')\widetilde{a} > \theta(\mathbb{Z} - nm(\lambda + \lambda'))$$

as required. If on the other hand $m\mu + n\mu' = 0$, then $0 = -(m\mu + n\mu')\widetilde{a} < \theta(\mathbb{Z} + nm(\lambda + \lambda')$, so

$$\lambda + \lambda' + \frac{(r_{nm} - r_n)}{n}\mu + \frac{(r_{nm} - r_m)}{m}\mu' > 0$$

since θ preserves the order on $X/\mathbb{Z}$ and $\frac{(r_{nm}-r_n)}{n}\mu + \frac{(r_{nm}-r_m)}{m}\mu' \in \mathbb{Z}$. Clearly then P is closed under addition.

The above argument shows that the ordering $<$ defines a positive cone, from which we conclude that it is a linear order which is respected by $+$, as required. Moreover, we can show that $\mathbb{Z}$ is still convex in $X[a]$. Take $\gamma \in X[a] \setminus \mathbb{Z}$ and $\gamma > 0$. This element is of the form $\gamma = \lambda + \frac{(x-r_n)}{n}\mu > 0$. Now if $\mu = 0$ then $\gamma = \lambda \in X$, and since $\mathbb{Z}$ is convex in X we therefore have that $\gamma > \mathbb{Z}$. On the other hand, if $\mu \neq 0$ we know that $\mu\widetilde{a} > \theta(\mathbb{Z} - n\lambda) \in \widetilde{\Gamma}$. In this case let $m \in \mathbb{Z}$ be arbitrary, then $\gamma + m = \lambda + m + \frac{(x-r_n)}{n}\mu$ and $\mu \neq 0$, hence $\theta(\mathbb{Z} + n(\lambda + m)) = \theta(\mathbb{Z} - n\lambda - nm) = \theta(\mathbb{Z} - n\lambda) < \mu\widetilde{a}$. For arbitrary $m \in \mathbb{Z}$ we therefore find that $\gamma + m > 0$, from which $\gamma > \mathbb{Z}$ follows, and so $\mathbb{Z}$ is clearly convex in $X[a]$.

The preceding results tell us that $X[a]$ has a discrete linear ordering respected by $+$. The remaining Presburger axioms follow in an analogous way to the construction in definition 2.2.3, from which we find that $X[a]$ is a Presburger group and hence that the residue map ϱ is defined on it. In particular we note that $\varrho(r_n + \frac{(x-r_n)}{n}n) = r$ and so we can set $a = r_n + \frac{(x-r_n)}{n}n \in X[a]$.

Define $\widehat{\theta}\colon X[a]/\mathbb{Z} \to \left\langle \widetilde{X}, \widetilde{a} \right\rangle$ to be the map

$$\widehat{\theta}\left(\mathbb{Z} + \lambda + \frac{(x - r_n)}{n}\mu\right) = \theta(\mathbb{Z} + \lambda) + \frac{\widetilde{a}\mu}{n},$$

which is possible since $\widetilde{a} \in \widetilde{X}$ where $\widetilde{X}$ is divisible. There are a number of things which it is necessary to check with regard to $\widehat{\theta}$.

To see that it is well defined, suppose that $\mathbb{Z} + \lambda + \frac{(x-r_n)}{n}\mu = \mathbb{Z} + \lambda' + \frac{(x-r_m)}{m}\mu'$. Then

$$\lambda + \frac{(x - r_n)}{n}\mu = \lambda' + \frac{(x - r_m)}{m}\mu' + c$$

where $c \in \mathbb{Z}$. By our definition we know that $m\mu = n\mu'$ and $nm(\lambda' + c) - nr_m\mu' = nm\lambda - mr_m\mu$. So

$$nm(\lambda' + c) = nm\lambda + (nr_m\mu' - mr_n\mu). \tag{3.2}$$

Assume first that $\lambda' + c \neq 0$. Then the left-hand side of (3.2) is a multiple of nm, so clearly $(nr_m\mu' - mr_n\mu)$ must also be, thus allowing us to divide through to get

$$\lambda' + c = \lambda + \frac{nr_m\mu' - mr_n\mu}{nm}.$$

This provides us with the result that $\mathbb{Z}+\lambda' = \mathbb{Z}+\lambda$ since both $c \in \mathbb{Z}$ and $\frac{nr_m\mu' - mr_n\mu}{nm} \in \mathbb{Z}$. Alternatively if $\lambda'+c = 0$ then $nm\lambda = mr_n\mu - nr_m\mu' \in \mathbb{Z}$, so $\mathbb{Z}+\lambda = \mathbb{Z}+0 = \mathbb{Z}+\lambda'+c = \mathbb{Z} + \lambda'$. In each case then, we have $\mathbb{Z} + \lambda' = \mathbb{Z} + \lambda$, and so $\theta(\mathbb{Z} + \lambda') = \theta(\mathbb{Z} + \lambda)$.

Moreover, since $m\mu = n\mu'$ we have that

$$\frac{\widetilde{a}\mu}{n} = \frac{\widetilde{a}\mu'}{m},$$

so

$$\begin{aligned}
\widehat{\theta}\left(\mathbb{Z} + \lambda + \frac{(x - r_n)}{n}\mu\right) &= \theta(\mathbb{Z} + \lambda) + \frac{\widetilde{a}\mu}{n} \\
&= \theta(\mathbb{Z} + \lambda') + \frac{\widetilde{a}\mu'}{m} \\
&= \widehat{\theta}\left(\mathbb{Z} + \lambda' + \frac{(x - r_m)}{m}\mu' + c\right).
\end{aligned}$$

We see from this that $\widehat{\theta}$ is well-defined.

It is also necessary that $\widehat{\theta}$ be an order-preserving isomorphism. So we start by showing that it is injective, taking

$$\widehat{\theta}\left(\mathbb{Z} + \lambda + \frac{(x - r_n)}{n}\mu\right) = \widehat{\theta}\left(\mathbb{Z} + \lambda' + \frac{(x - r_m)}{m}\mu'\right).$$

This is equivalent to saying that

$$\theta(\mathbb{Z}+\lambda)+\frac{\widetilde{a}\mu}{n}=\theta(\mathbb{Z}+\lambda')+\frac{\widetilde{a}\mu'}{m}.$$

Since $\widetilde{a}\notin\left\langle\widetilde{X}\right\rangle$ and θ is an isomorphism we therefore know that $\theta(\mathbb{Z}+\lambda)=\theta(\mathbb{Z}+\lambda')$ if and only if $\mathbb{Z}+\lambda=\mathbb{Z}+\lambda'$. This is equivalent to saying that $\lambda'=\lambda+s$ for some $s\in\mathbb{Z}$. It is also clear that

$$\frac{\widetilde{a}\mu}{n}=\frac{\widetilde{a}\mu'}{m} \quad \text{if and only if} \quad \mu m=\mu' n.$$

Now $r_{nm}\equiv r_n \pmod{n}$ and $r_n m\equiv r_m \pmod{m}$, so n divides $r_{nm}-r_n$ and m divides $r_{nm}-r_m$. Thus we may define $t\in\mathbb{Z}$ to be

$$t=\frac{(r_{nm}-r_m)}{m}\mu'-\frac{(r_{nm}-r_n)}{n}\mu+s.$$

Then

$$\begin{aligned}\lambda+\frac{(x-r_n)}{n}\mu+t&=\lambda+s+\frac{(x-r_n)m\mu}{nm}+\frac{(r_{nm}-r_m)n\mu'}{nm}-\frac{(r_{nm}-r_n)m\mu}{nm}\\&=\lambda'+\frac{xm\mu-r_nm\mu+r_{nm}n\mu'-r_mn\mu'-r_mn\mu'-r_{nm}m\mu+r_nm\mu}{nm}\\&=\lambda'+\frac{xn\mu'-r_mn\mu'+r_{nm}n\mu'-r_{nm}n\mu'}{nm}\\&=\lambda'+\frac{(x-r_m)}{m}\mu',\end{aligned}$$

so that

$$\mathbb{Z}+\lambda+\frac{(x-r_n)}{n}\mu=\mathbb{Z}+\lambda'+\frac{(x-r_m)}{m}\mu',$$

giving the result that $\widehat{\theta}$ is indeed injective.

For surjectivity, consider an arbitrary element $\widetilde{\gamma}\in\left\langle\widetilde{X},\widetilde{a}\right\rangle$. Then $\widetilde{\gamma}=\widetilde{\lambda}+q\widetilde{a}$ for $\widetilde{\lambda}\in\widetilde{X}$ and $q=\frac{\mu}{n}\in\mathbb{Q}$ with $\mu\in\mathbb{Z},n\in\mathbb{N}$ and $\theta\colon X/\mathbb{Z}\cong\widetilde{X}$ is an isomorphism, so $\lambda=\theta^{-1}(\widetilde{\lambda})\in X$. We can therefore set

$$\gamma=\lambda+\frac{(x-r_n)}{n}\mu \quad \text{to get} \quad \begin{aligned}\widehat{\theta}(\gamma)&=\widehat{\theta}\left(\lambda+\frac{(x-r_n)}{n}\mu\right)\\&=\theta(\lambda)+\frac{\widetilde{a}\mu}{n}\\&=\widetilde{\lambda}+q\widetilde{a}\\&=\widetilde{\gamma}.\end{aligned}$$

We can therefore see that $\widehat{\theta}$ is surjective.

It is also a homomorphism, since

$$\begin{aligned}
\widehat{\theta}\left(\mathbb{Z}+\lambda+ \frac{(x-r_n)}{n}\mu\right) &+ \widehat{\theta}\left(\mathbb{Z}+\lambda+\frac{(x-r_n)}{n}\mu\right) \\
&= \theta(\mathbb{Z}+\lambda)+\frac{\widetilde{a}\mu}{n}+\theta(\mathbb{Z}+\lambda')+\frac{\widetilde{a}\mu'}{m} \\
&= \theta(\mathbb{Z}+\lambda+\lambda')+\frac{\widetilde{a}(m\mu+n\mu')}{nm} \\
&= \widehat{\theta}\left(\mathbb{Z}+\lambda+\lambda'+\frac{(x-r_{nm})}{nm}(m\mu+n\mu')\right) \\
&= \widehat{\theta}\left(\mathbb{Z}+\lambda+\lambda'+\frac{(x-r_{nm})}{nm}(m\mu+n\mu')+\frac{(r_{nm}-r_n)}{n}\mu+\frac{(r_{nm}-r_m)}{m}\mu'\right) \\
&= \widehat{\theta}\left(\mathbb{Z}+\lambda+\frac{(x-r_n)}{n}\mu+\mathbb{Z}+\lambda+\frac{(x-r_n)}{n}\mu\right).
\end{aligned}$$

Finally, to show that it preserves order, suppose that

$$\mathbb{Z}+\lambda+\frac{(x-r_n)}{n}\mu>0.$$

Then

$$\widehat{\theta}(\mathbb{Z}+\lambda+\frac{(x-r_n)}{n}\mu)=\theta(\mathbb{Z}+\lambda)+\frac{\mu\widetilde{a}}{n}.$$

If $\mu=0$ then $\lambda>0$ so $\theta(\mathbb{Z}+\lambda)>0$ as θ respects order. If $\mu\neq 0$ then $\mu\widetilde{a}>\theta(\mathbb{Z}-n\lambda)$, so $\theta(\mathbb{Z}+n\lambda)+\mu\widetilde{a}>0$ and since $\widetilde{\Gamma}$ is divisible and θ is a homomorphism we therefore obtain the result that $\theta(\mathbb{Z}+\lambda)+\frac{\mu\widetilde{a}}{n}>0$, as required.

So finally we see that $\widehat{\theta}$ is indeed an order-preserving isomorphism.

In order to provide us with a contradiction to the maximality of (X,θ) it remains to show that the diagram

$$\begin{array}{ccccc}
X[a] & \longrightarrow & X[a]/\mathbb{Z} & \xrightarrow{\theta} & \left\langle \widetilde{X},\widetilde{a}\right\rangle \\
\downarrow \varrho & & & & \downarrow \widetilde{\varrho} \\
\widehat{\mathbb{Z}} & & \longrightarrow & & \widehat{\mathbb{Z}}/\mathbb{Z}
\end{array}$$

commutes. *i.e.* we must show that for arbitrary $\lambda+\frac{(x-r_n)}{n}\mu\in X[a]$,

$$\eta'\cdot\varrho\left(\lambda+\frac{(x-r_n)}{n}\mu\right)=\widetilde{\varrho}\cdot\widehat{\theta}\cdot\eta\left(\lambda+\frac{(x-r_n)}{n}\mu\right),$$

where $\eta\colon X[a]\to X[a]/\mathbb{Z}$ and $\eta'\colon\widehat{\mathbb{Z}}\to\widehat{\mathbb{Z}}/\mathbb{Z}$ are the natural quotient maps.

We first show that $\widetilde{\varrho}\left(\frac{\widetilde{a}\mu}{n}\right)=\mathbb{Z}+\varrho\left(\frac{(x-r_n)}{n}\mu\right)$. In order to do this we must consider what we mean by $\widetilde{\varrho}\left(\frac{\widetilde{a}\mu}{n}\right)$. It is equivalent to saying that for some $\widetilde{y}\in\left\langle\widetilde{X},\widetilde{a}\right\rangle$, $n\widetilde{\varrho}(\widetilde{y})=$

$\widetilde{\varrho}(\widetilde{a}\mu) = \mu\widetilde{\varrho}(\widetilde{a}) = \mu(\mathbb{Z}+r) = \mathbb{Z}+\mu r$. By $\mathbb{Z}+\varrho\left(\frac{(x-r_n)}{n}\mu\right)$ we mean that for some $y \in X[a]$ we have $\mathbb{Z} + n\varrho(y\mu) = \mathbb{Z} + \varrho((x-r_n)\mu) = \mathbb{Z} + \varrho(x\mu) - \varrho(r_n\mu) = \mathbb{Z} + \mu\varrho(x) = \mathbb{Z} + \mu r$. So for these $\widetilde{y} \in \left\langle \widetilde{X}, \widetilde{a} \right\rangle, y \in X[a]$ we have $n\widetilde{\varrho}(\widetilde{y}\mu) = \mathbb{Z}+n\varrho(y\mu)$, so $\widetilde{\varrho}(\widetilde{y}\mu) = \mathbb{Z}+\varrho(y\mu)$ and hence $\widetilde{\varrho}\left(\frac{\widetilde{a}\mu}{n}\right) = \mathbb{Z} + \varrho\left(\frac{(x-r_n)}{n}\mu\right)$ as required.

To return to our main problem,

$$\begin{aligned}
\varrho \cdot \eta'\left(\lambda + \frac{(x-r_n)}{n}\mu\right) &= \eta'\left(\varrho(\lambda) + \varrho\left(\frac{(x-r_n)}{n}\mu\right)\right) \\
&= \mathbb{Z} + \varrho(\lambda) + \mathbb{Z} + \varrho\left(\frac{(x-r_n)}{n}\mu\right) \\
&= \widetilde{\varrho}(\theta(\mathbb{Z}+\lambda)) + \widetilde{\varrho}\left(\frac{\widetilde{a}\mu}{n}\right) \\
&= \widetilde{\varrho}\left(\theta(\mathbb{Z}+\lambda) + \frac{\widetilde{a}\mu}{n}\right) \\
&= \widetilde{\varrho}\left(\widehat{\theta}\left(\mathbb{Z}+\lambda + \frac{(x-r_n)}{n}\mu\right)\right) \\
&= \eta \cdot \widehat{\theta} \cdot \widetilde{\varrho}\left(\lambda + \frac{(x-r_n)}{n}\mu\right),
\end{aligned}$$

which is what was required. So we may conclude that $(X,\theta) < (X[a],\widehat{\theta})$ and since $(X[a],\widehat{\theta}) \in \mathscr{P}$, we achieve a contradiction to the maximality of (X,θ). Hence no such $\widetilde{a} \in \widetilde{\Gamma}$ can exist and $\widetilde{X} = \widetilde{\Gamma}$. □

Proposition 3.1.2 and theorem 3.2.1 can be quite powerful when used together, since they allow us to swap between Presburger groups and divisible ordered abelian groups with relative ease. This therefore allows us to utilise the useful divisibility properties of the latter in a constructive manner. In view of the previous theorem we embrace the following notation.

Notation. We write $\widetilde{A}$ instead of $A/\mathbb{Z}$ for any set A of Presburger elements. Hence $\widetilde{\Gamma} = \Gamma/\mathbb{Z}$.

The correlation between Presburger groups and divisible ordered abelian groups will become particularly useful when we start to construct automorphisms of Presburger arithmetic later on, and we will use the proposition which follows the ensuing definition extensively.

Definition 3.2.2. Let $\alpha\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ be an automorphism of $\widetilde{\Gamma}$ as an ordered divisible abelian group so that

$$\widetilde{\varrho}(\alpha(a)) = \widetilde{\varrho}(a) \quad \text{for all } a \in \widetilde{\Gamma}.$$

Then we say that the map $\alpha\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ **preserves residues** and that α is a **residue-automorphism**.

Proposition 3.2.3. Suppose $\alpha\colon \Gamma/\mathbb{Z} \to \Gamma/\mathbb{Z}$ is a residue-automorphism. Then α lifts to an automorphism of Γ:

$$\widehat{\alpha}\colon \Gamma \to \Gamma.$$

Proof. The restriction $\varrho\restriction_{\mathbb{Z}+x}$ to a coset is an injection $\mathbb{Z} + x \to \widehat{\mathbb{Z}}$, hence for each $\gamma \in \mathbb{Z} + x$ there is a unique element $\gamma' \in \alpha(\mathbb{Z} + x)$ such that $\varrho(\gamma) = \varrho(\gamma')$. From this we can immediately construct a well-defined and bijective lifting of α to $\widehat{\alpha}$. We must show that $\widehat{\alpha}$ preserves ordering. So suppose we have $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1 < \gamma_2$. Then if $\gamma_1/\mathbb{Z} < \gamma_2/\mathbb{Z}$ we immediately have that $\widehat{\alpha}(\gamma_1) < \widehat{\alpha}(\gamma_2)$ since α is order preserving. If on the other hand $\gamma_1 + n = \gamma_2$ for some $n \in \mathbb{Z}_{>0}$ then $\varrho(\gamma_1 + n) = \varrho(\gamma_2)$ and hence $\widehat{\alpha}(\gamma_1 + n) = \widehat{\alpha}(\gamma_2)$. We then have $\widehat{\alpha}(\gamma_1) + n = \widehat{\alpha}(\gamma_2)$ and hence $\widehat{\alpha}(\gamma_1) < \widehat{\alpha}(\gamma_2)$ as required. $\square$

Again we introduce some notation which we hope will simplify our use of divisible ordered abelian groups later on. Although we will not make use of this notation until chapter 7, we present it here to avoid separating it from the notation already established in this chapter.

Notation. Suppose Γ is a Presburger group with ϱ its residue map. Then we write $\widetilde{\varrho}\colon \widetilde{\Gamma} \to \widehat{\mathbb{Z}}/\mathbb{Z}$ to mean the induced map

$$\varrho/\mathbb{Z}\colon \Gamma/\mathbb{Z} \to \widehat{\mathbb{Z}}/\mathbb{Z}.$$

Definition 3.2.4. If Γ is a Presburger group we define $\mathrm{Aut}(\Gamma)$ to be the set of automorphisms of Γ and $\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})$ to be the set of residue-automorphisms of $\widetilde{\Gamma}$.

Chapter 4

Quantifier Elimination

4.1 Preliminaries

In this section we will be providing a proof of the significant 'Quantifier Elimination' result for Presburger Arithmetic. As a consequence of this result we find that any logically finite statement about a Presburger group $(\Gamma, +, <, 0, 1)$ is logically equivalent to a Δ_1 statement about that group, *i.e.* to a $\exists^{\leq}$ or $\forall^{\leq}$ statement containing only bounded quantifiers. This pivotal result ensures that we are able to say far more about Presburger Arithmetic than we would otherwise be able to.

In order to prove the Quantifier Elimination result, we must first establish a number of definitions and lemmas. Throughout this chapter we shall assume Γ to be a model of Presburger Arithmetic.

Notation. If Γ is a Presburger group, we use the notation $\bar{x} \in \Gamma$ to mean that $\bar{x} = (x_0, x_1, \ldots, x_i, \ldots)_{i<\alpha}$ is a sequence of elements $x_i \in \Gamma$ indexed by an ordinal α. For the case when $\bar{x} = (x_0, x_1, \ldots, x_n)$ where $n \in \mathbb{N}$ we say that $\bar{x}$ has finite length and set $\text{len}(\bar{a}) = n$.

To study countable Presburger groups, it is only necessary to consider sequences $\bar{x}$ of finite length, so for this and subsequent sections we will take $\bar{x} \in \Gamma$ to mean $\bar{x}$ has finite length $n \in \mathbb{N}$, unless explicitly stated otherwise.

Suppose we have a pair of Presburger groups Γ_1 and Γ_2 and a monomorphism $\alpha\colon \Gamma_1 \to \Gamma_2$. If $\bar{x} \in \Gamma_1$ and $\bar{y} = \alpha(\bar{x}) \in \Gamma_2$, then the following properties are obvious:

1. $\Gamma_1 \models \lambda_0 + \sum_i \lambda_i x_{i+1} > 0 \Longleftrightarrow \Gamma_2 \models \lambda_0 + \sum_i \lambda_i y_{i+1} > 0$ for all $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{Z}$

2. $\Gamma_1 \models x_i \equiv r \pmod{n} \iff \Gamma_2 \models y_i \equiv r \pmod{n}$ for all $i, 0 \leq r < n$ and $n \in \mathbb{N}, n \geq 1$.

A special case of (1) occurs when $\alpha(1) = 1$.

Now suppose Γ_1, Γ_2 are Presburger groups and $\bar{x} \in \Gamma_1, \bar{y} \in \Gamma_2$ are arbitrary. We write

$$\bar{x} \equiv_0 \bar{y}$$

if $\bar{x}, \bar{y}$ have the same length and (1),(2) above hold. If we define $\langle 1, x \rangle = \{ \lambda_0 + \sum_i \lambda_i x_i : \lambda_i \in \mathbb{Z}\}$ then it immediately follows that

$$\bar{x} \equiv_0 \bar{y} \quad \Rightarrow \quad \langle 1, \bar{x} \rangle \cong \langle 1, \bar{y} \rangle$$

where $\langle 1, \bar{x} \rangle$ and $\langle 1, \bar{y} \rangle$ are considered as ordered groups, the ordering being induced from the ordering of $\bar{x}$ and $\bar{y}$ in Γ_1 and Γ_2 respectively.

We also need to define a finite version of $\equiv_0$:

Definition 4.1.1. If $\bar{x} \in \Gamma_1, \bar{y} \in \Gamma_2$, and $n \in \mathbb{N}$ is fixed, we write

$$\bar{x} \underset{0,n}{\sim} \bar{y}$$

if and only if $\bar{x}, \bar{y}$ have the same length and

1. $\lambda_0 + \sum_i \lambda_i x_{i+1} > 0 \iff \lambda_0 + \sum_i \lambda_i y_{i+1} > 0$ for all $\lambda_0, \ldots, \lambda_n \in \mathbb{Z}$ with $|\lambda_i| < n$ for all i.

2. $x_i \equiv r \pmod{s} \iff y_i \equiv r \pmod{s}$ for all $i, 0 \leq r < s, 1 \leq s < n$.

This definition can then be extended by induction, using the idea of back-and-forth:

Definition 4.1.2. If $\bar{x} \in \Gamma_1, \bar{y} \in \Gamma_2$, and $n \in \mathbb{N}$ is fixed, we write

$$\bar{x} \underset{r+1,n}{\sim} \bar{y}$$

to mean that

$$\forall u \in \Gamma_1 \, \exists v \in \Gamma_2 (\bar{x}, u \underset{r,n}{\sim} \bar{y}, v)$$

and that

$$\forall v \in \Gamma_2 \, \exists u \in \Gamma_1 (\bar{x}, u \underset{r,n}{\sim} \bar{y}, v).$$

Lemma 4.1.3. For all $n \geq 1$ in $\mathbb{N}, x \in \Gamma$ and $r \in \{0, 1, \ldots, n-1\}$ there exists $s \in \{0, 1, \ldots, n-1\}$ such that $x + s \equiv r \pmod{n}$.

Proof. By the Presburger axiom $\mathbf{A_3}$ we know that $x \equiv t \pmod{n}$ for some $t < n$. We also know that we may find some $s < n$ such that $t + s \equiv r \pmod{n}$. Hence $x + s \equiv t + s \equiv r \pmod{n}$. ☐

Definition 4.1.4. We define $\left\lceil \frac{y}{n} \right\rceil$ to be the unique z such that $nz = y + r$ where $y \equiv -r \pmod{n}$ and $0 \leq r < n$.

It is instructive to note that in accordance with the above definition, we have that $\left\lceil \frac{y}{n} \right\rceil = \frac{y+r}{n}$ and $\frac{y}{n} \leq \left\lceil \frac{y}{n} \right\rceil < \frac{y+n}{n}$.

Definition 4.1.5. We define $\left\lfloor \frac{y}{n} \right\rfloor$ to be the unique z such that $nz = y - r$ where $y \equiv r \pmod{n}$ and $0 \leq r < n$.

Again we see that from this definition $\left\lfloor \frac{y}{n} \right\rfloor = \frac{y-r}{n}$ and $\frac{y-n}{n} < \left\lfloor \frac{y}{n} \right\rfloor \leq \frac{y}{n}$.

Lemma 4.1.6. For $n \geq 1$ in $\mathbb{N}$ we have that:

$$nx \leq y \iff x \leq \left\lfloor \frac{y}{n} \right\rfloor, \tag{4.1}$$

and

$$nx \geq y \iff x \geq \left\lceil \frac{y}{n} \right\rceil. \tag{4.2}$$

Proof. For equation (4.1) suppose $nx \leq y$. Then

$$\begin{array}{rrcll} & nx & < & y + 1 & \\ \Rightarrow & nx & < & y + n - (n-1) & \\ \Rightarrow & nx & < & y + n - r & \text{since } 0 \leq r < n \\ \Rightarrow & x & < & \frac{y-r}{n} + 1 & \\ \Rightarrow & x & \leq & \left\lfloor \frac{y}{n} \right\rfloor. & \end{array}$$

The reverse direction follows almost immediately, since if

$$x \leq \left\lfloor \frac{y}{n} \right\rfloor = \frac{y-r}{n}$$

then $xn \leq y - r$ and since $r \geq 0$ we have that $xn \leq y$ as required.

The proof for equation (4.2) is practically identical to this. ☐

Lemma 4.1.7. Let Γ, Γ' be models of Presburger arithmetic, $\bar{x} \in \Gamma, n \in \mathbb{N}$ and $\nu, \mu, \mu', \bar{\lambda}, \bar{\lambda}' \in \mathbb{Z}$ with $0 < \mu, \mu' \leq n$ and $|\bar{\lambda}|, |\bar{\lambda}'| \leq n$. Suppose

$$\left\lfloor \frac{\lambda_0 + \sum_i \lambda_i x_i}{\mu} \right\rfloor \geq \left\lceil \frac{\lambda'_0 + \sum_i \lambda'_i x_i}{\mu'} \right\rceil + \nu, \tag{4.3}$$

then there exists $\bar{\zeta} \in \mathbb{Z}$ such that

$$\zeta_0 + \sum_i \zeta_i x_i > 0, |\bar{\zeta}| \leq n^2(|\nu| + 4)$$

and for which, whenever $\bar{y} \in \Gamma'$ satisfies

$$\zeta_0 + \sum_i \zeta_i y_i > 0 \quad \text{and} \quad x_i \equiv y_i \pmod{\mu}, x_i \equiv y_i \pmod{\mu'} \text{ for all } i,$$

we have

$$\left\lfloor \frac{\lambda_0 + \sum_i \lambda_i y_i}{\mu} \right\rfloor \geq \left\lceil \frac{\lambda'_0 + \sum_i \lambda'_i y_i}{\mu'} \right\rceil + \nu. \tag{4.4}$$

Proof. We let $r, s \in \mathbb{Z}$ be such that $\lambda_0 + \sum_i \lambda_i x_i \equiv r \pmod{\mu}, \lambda'_0 + \sum_i \lambda'_i x_i \equiv -s \pmod{\mu'}$ with $0 \leq r < \mu$ and $0 \leq s < \mu'$. By considering the equation (4.3) we see that

$$\begin{aligned}
&\left\lfloor \frac{\lambda_0 + \sum_i \lambda_i x_i}{\mu} \right\rfloor \geq \left\lceil \frac{\lambda'_0 + \sum_i \lambda'_i x_i}{\mu'} \right\rceil + \nu \\
&\quad\iff \mu\mu' \left\lfloor \frac{\lambda_0 + \sum_i \lambda_i x_i}{\mu} \right\rfloor \geq \mu\mu' \left\lceil \frac{\lambda'_0 + \sum_i \lambda'_i x_i}{\mu'} \right\rceil + \mu\mu'\nu \\
&\quad\iff \mu'(\lambda_0 + \sum_i \lambda_i x_i - r) \geq \mu(\lambda'_0 + \sum_i \lambda'_i x_i + s) + \mu\mu'\nu \\
&\quad\iff \sum_i (\mu'\lambda_i - \mu\lambda'_i)x_i + \mu'\lambda_0 - \mu' r - \mu\lambda'_0 - \mu s - \mu\mu'\nu + 1 > 0.
\end{aligned}$$

Clearly by setting $\zeta_i = \mu'\lambda_i - \mu\lambda'_i$ and $\zeta_0 = \mu'\lambda_0 - \mu' r - \mu\lambda'_0 - \mu s - \mu\mu'\nu + 1$ we have $|\zeta_i| < 2n^2$ and $|\zeta_0| < 4n^2 + n^2\nu$ which gives us the required result. □

4.2 The quantifier elimination theorem

Theorem 4.2.1 (Elimination of Quantifiers). If Γ_1 and Γ_2 are Presburger groups then for every $n \in \mathbb{N} \setminus \{0\}$ there is an $n' \in \mathbb{N}$ such that

$$\forall \bar{x} \in \Gamma_1 \, \forall \bar{y} \in \Gamma_2 \quad \bar{x} \underset{0,n'}{\sim} \bar{y} \quad \Rightarrow \quad \bar{x} \underset{1,n}{\sim} \bar{y}.$$

Proof. Suppose Γ_1, Γ_2 are as in the theorem and that $n \in \mathbb{N} \setminus \{0\}$. Take $n' = n^2(l + 4)$ where $l = \operatorname{lcm}(1, 2, \ldots, n-1)$. Now suppose that $\bar{x} \in \Gamma_1$ and $\bar{y} \in \Gamma_2$ are such that $\bar{x} \underset{0,n'}{\sim} \bar{y}$. Taking an arbitrary $z' \in \Gamma_1$ we hope to find some $y' \in \Gamma_2$ which satisfies $\bar{x}, x' \underset{0,n}{\sim} \bar{y}, y'$. Once we have repeated this process in the reverse direction we will be done.

So we take $\bar{x}'$ as suggested above and consider all true order relations which can be expressed either in the form

$$\lambda_0 + \sum_i \lambda_i x_i + \mu x' > 0 \quad \text{where} \quad |\bar{\lambda}|, |\mu| < n', \mu \neq 0,$$

or as the negations of these, which amount to relations of the form

$$\lambda_0 + \sum_i \lambda_i x_i + \mu x' \leq 0 \quad \text{where} \quad |\bar{\lambda}|, |\mu| < n', \mu \neq 0.$$

By virtue of lemma 4.1.6 we can re-express all of these inequalities as upper or lower bounds, in the following ways:

$$\begin{aligned} \mu x' &\leq \lambda_0 + \sum_i \lambda_i x_i &\iff& \quad x' \leq \left\lfloor \frac{\lambda_0 + \sum_i \lambda_i x_i}{\mu} \right\rfloor, \\ \mu x' &\geq \lambda_0 + \sum_i \lambda_i x_i &\iff& \quad x' \geq \left\lceil \frac{\lambda_0 + \sum_i \lambda_i x_i}{\mu} \right\rceil, \\ & & & \text{where } 0 < \mu \leq n \text{ and } |\bar{\lambda}| \leq n. \end{aligned}$$

We set v to be the maximum of these lower bounds, and v' to be the minimum of all of the upper bounds. Hence we have, for some fixed $0 < \mu, \mu' \leq n$ and $|\bar{\lambda}|, |\bar{\lambda}'| \leq n$,

$$v = \left\lceil \frac{\lambda_0 + \sum_i \lambda_i x_i}{\mu} \right\rceil \quad \text{and} \quad v' = \left\lfloor \frac{\lambda'_0 + \sum_i \lambda'_i x_i}{\mu'} \right\rfloor.$$

We also set w and w' to be the corresponding maximum upper and minimum lower bounds for $\bar{y}$ in Γ_2:

$$w = \left\lceil \frac{\lambda_0 + \sum_i \lambda_i y_i}{\mu} \right\rceil \quad \text{and} \quad w' = \left\lfloor \frac{\lambda'_0 + \sum_i \lambda'_i y_i}{\mu'} \right\rfloor.$$

Clearly from lemma 4.1.7 we see that $v \leq v' \quad \Rightarrow \quad w \leq w'$. We also have that $v \leq x' \leq w'$. To complete this part of the proof we must therefore find y' such that $w \leq y' \leq w'$ and for which $x' \equiv y' \pmod{m}$ for all $m < n$. Two cases arise, which we consider separately.

Case 1. $\quad x' = v + i$ for some $i < l$.

Clearly we must have $x' = v + i \leq v'$, so by lemma 4.1.7 there exists $\bar{\zeta} \in \mathbb{Z}$ with $\bar{\zeta} \leq n'$ and with the properties given in that lemma. But by our assumptions we know that $\bar{y}$ satisfies

$$\zeta_0 + \sum_i \zeta_i y_i > 0 \quad \text{and} \quad x_i \equiv y_i \pmod{\mu}, x_i \equiv y_i \pmod{\mu'} \text{ for all } i$$

and hence we have $w+i \le w'$. We choose $y' = w+i$. Since this clearly satisfies that $w \le y' \le w'$, it suffices to show that it has the same residues as x', up to $\bmod n$.

Suppose that $x_i \equiv r_i \pmod{\mu' n}$, where $(2 \le m < n$ and $0 \le r_i < \mu' m)$. Suppose further that

$$\begin{aligned} & \lambda_0' + \sum_i \lambda_i' x_i \equiv -s \pmod{\mu'} && (0 \le s < \mu'), \\ \Rightarrow\quad & \lambda_0' + \sum_i \lambda_i' r_i + s \equiv 0 \pmod{\mu'}, \\ \Rightarrow\quad & \lambda_0' + \sum_i \lambda_i' r_i + s = \mu' && \text{for some } a \in \mathbb{Z}. \end{aligned}$$

Also,

$$\begin{aligned} \left\lceil \frac{\lambda_0' + \sum_i \lambda_i' x_i}{\mu'} \right\rceil &= \frac{1}{\mu'} \left(\lambda_0' + \sum_i \lambda_i' x_i + s \right), \\ &= \frac{1}{\mu'} \left(\lambda_0' + \sum_i \lambda_i' r_i + s \right) + \frac{1}{\mu'} \left(\lambda_0' + \sum_i \lambda_i' \mu' m Z_i \right) \\ &\qquad \text{with } x_i = r_i + \mu' m Z_i, \\ &\equiv a \pmod{m}. \end{aligned}$$

But $\mu' m < n^2 \le n'$ so $y_i \equiv r_i \pmod{\mu' m}$. Hence

$$\left\lceil \frac{\lambda_0' + \sum_i \lambda_i' y_i}{\mu'} \right\rceil \equiv a \pmod{m},$$

as required.

Case 2. $x' \ge v + l$.

Suppose $x' \equiv r \pmod{l}$. Then for some $y' = v + i$ where $i < l$, we have $y' \equiv r \pmod{l}$. But then $x' \equiv y' \pmod{m}$ for all $m < n$ and $w \le y' \le w'$ as required. □

It is worth drawing attention to the fact — as can be seen in the early part of the proof — that the value which is required of n' is *independent of the sequences* $\bar{x}$ *and* $\bar{y}$.

Corollary 4.2.2. If Γ_1 and Γ_2 are Presburger groups then for each $r \in \mathbb{N}$ and each $n \in \mathbb{N}$ there exists $n'' \in \mathbb{N}$ such that

$$\forall \bar{x} \in \Gamma_1 \, \forall \bar{y} \in \Gamma_2 \quad \bar{x} \underset{0,n''}{\smile} \bar{y} \quad \Rightarrow \quad \bar{x} \underset{r,n}{\smile} \bar{y}.$$

Proof. We prove this by induction on r.

The base case, $r = 1$, follows directly from the previous theorem 4.2.1. For $r > 1$, by using our inductive hypothesis we may find $n' \in \mathbb{N}$ such that

$$\forall \bar{x}, x' \, \forall \bar{y}, y' \quad \bar{x}, x' \underset{0,n'}{\smile} \bar{y}, y' \quad \Rightarrow \quad \bar{x}, x' \underset{r-1,n}{\smile} \bar{y}, y'. \tag{4.5}$$

By theorem 4.2.1 we may infer from this that, for some $n'' \in \mathbb{N}$ we have

$$\forall \bar{x} \, \forall \bar{y} \quad \bar{x} \underset{0,n''}{\smile} \bar{y} \quad \Rightarrow \quad \bar{x} \underset{1,n'}{\smile} \bar{y}. \tag{4.6}$$

So, $\forall \bar{x} \, \forall \bar{y}$

$$\begin{aligned} \bar{x} \underset{0,n''}{\smile} \bar{y} \quad &\Rightarrow \quad \bar{x} \underset{1,n'}{\smile} \bar{y} && \text{(by 4.6)}, \\ &\Rightarrow \quad \forall x' \, \exists y' \quad \bar{x}, x' \underset{0,n'}{\smile} \bar{y}, y', && \text{(by definition of } \underset{1,n'}{\smile}), \\ &\Rightarrow \quad \forall x' \, \exists y' \quad \bar{x}, x' \underset{r-1,n}{\smile} \bar{y}, y' && \text{(by 4.5)}. \end{aligned}$$

Similarly, we have, $\forall \bar{x} \, \forall \bar{y}$

$$\begin{aligned} \bar{x} \underset{0,n''}{\smile} \bar{y} \quad &\Rightarrow \quad \bar{x} \underset{1,n'}{\smile} \bar{y} && \text{(by 4.6)}, \\ &\Rightarrow \quad \forall y' \, \exists x' \quad \bar{x}, x' \underset{0,n'}{\smile} \bar{y}, y' && \text{(by definition of } \underset{1,n'}{\smile}), \\ &\Rightarrow \quad \forall y' \, \exists x' \quad \bar{x}, x' \underset{r-1,n}{\smile} \bar{y}, y' && \text{(by 4.5)}. \end{aligned}$$

Hence

$$\bar{x} \underset{0,n''}{\smile} \bar{y} \quad \Rightarrow \quad \bar{x} \underset{r,n}{\smile} \bar{y}$$

as required. □

Corollary 4.2.3. If $\bar{x} \in \Gamma_1, \bar{y} \in \Gamma_2$ are as in theorem 4.2.1, then

$$(\Gamma_1, \bar{x}) \equiv_0 (\Gamma_2, \bar{y}) \quad \Rightarrow \quad (\Gamma_1, \bar{x}) \underset{r,n}{\smile} (\Gamma_2, \bar{y})$$

for all $r, n \in \mathbb{N}$. In particular, $\Gamma_1 \underset{r,n}{\smile} \Gamma_2$ for all models of Presburger arithmetic Γ_1, Γ_2.

Proof. The first part is trivial since by observation it is clear that

$$(\Gamma_1, \bar{x}) \equiv_0 (\Gamma_2, \bar{y}) \quad \Rightarrow \quad (\Gamma_1, \bar{x}) \underset{0,n''}{\smile} (\Gamma_2, \bar{y})$$

for all $n'' \in \mathbb{N}$. The result then follows by corollary 4.2.2.

To show that $\Gamma_1 \underset{r,n}{\smile} \Gamma_2$ we show that $\Gamma_1 \equiv_0 \Gamma_2$. But again this is straightforward, since all that is meant by this is that

$$\Gamma_1 \models \lambda_0 > 0 \Longleftrightarrow \Gamma_2 \models \lambda_0 > 0 \text{ for all } \lambda_0 \in \mathbb{Z},$$

which is clearly the case. □

This last result tells us that the theory of Presburger arithmetic is complete. For if Γ_1 is a model of Presburger arithmetic and $\Gamma_1 \models \phi$ where ϕ is any sentence of our language, then we can convert ϕ into an equivalent sentence in prenex normal form called ϕ', say. Suppose ϕ' has r quantifiers and has constants with maximum value less than n. Then by the above result $\Gamma_1 \underset{r,n}{\sim} \Gamma_2$ so $\Gamma_1 \models \phi'$ if and only if $\Gamma_2 \models \phi'$. Hence every model of Presburger arithmetic must also have ϕ as a consequence.

Chapter 5

Types and Semi-Types

5.1 Preliminaries

Definition 5.1.1. A Presburger group **embedding** is an injective group homomorphism

$$h\colon \Gamma_1 \to \Gamma_2$$

which preserves both the identity 1 and the ordering $<$.

Definition 5.1.2. If $\Gamma_1 \subseteq \Gamma_2$ and the inclusion map $\Gamma_1 \to \Gamma_2$ is an embedding, then Γ_2 is said to be an **extension** of Γ_1.

If $x \in \Gamma, n \in \mathbb{N}$ and $x \equiv r \pmod{n}$ then $\frac{x-r}{n} \in \Gamma$. Hence we have the following definition:

Definition 5.1.3. If $A \subseteq \Gamma$, then for each $n \in \mathbb{N}, \bar{x} \in \mathbb{Z}, \bar{a} \in A$ with $a_i \equiv r_i \pmod{n}$, we have

$$x_0 + x_1\left(\frac{a_1 - r_1}{n}\right) + \cdots + x_n\left(\frac{a_n - r_n}{n}\right) \in \Gamma.$$

We denote the set of all such elements by $\mathrm{cl}(A)$.

Proposition 5.1.4. If $A \subseteq \Gamma$ then $\mathrm{cl}(A)$ is a Presburger group, and

$$A \subseteq \mathrm{cl}(A) \subseteq \Gamma.$$

Since all of the elements of $\mathrm{cl}(A)$ must be contained in every Presburger group containing A, it is clear that $\mathrm{cl}(A)$ is the smallest Presburger group containing A.

Definition 5.1.5. The extension $\Gamma_1 \subseteq \Gamma_2$ is **finite** (or **finitely generated**) if

$$\Gamma_2 = \mathrm{cl}(\Gamma_1 \cup A) \text{ for some finite set } A \subseteq \Gamma_2.$$

5.2 Basic formulas and types

Definition 5.2.1. Given a Presburger group Γ, a set $A \subseteq \Gamma$ and a sequence of variables $\bar{x} = (x_0, \ldots, x_j, \ldots)_{j<\alpha}$, a **basic formula in $\bar{x}$ over A** (or **with parameters from A**) is a formula of one of the following two kinds:

$$\zeta + \sum_i \gamma_i x_{j_i} > 0 \quad \text{where } \zeta \in \mathrm{cl}(A), \gamma_i \in \mathbb{Z}, \bar{\jmath} < \alpha,$$
$$\text{or} \quad x_i \equiv r \pmod{n} \quad \text{where } i < \alpha, 0 \leq r < n \in \mathbb{N}.$$

We denote a set of such formulas by using an expression such as $p(\bar{x})$ or $q(\bar{x})$.

Definition 5.2.2. Let Γ be some model with $A \subseteq \Gamma$, $\bar{x}$ a sequence of variables and $\bar{a} \in \Gamma$ a sequence of the same length as $\bar{x}$. Then the **type of $\bar{a}$ over** A, $\mathrm{tp}(\bar{a}/A)$, is defined to be the set of all basic formulas in $\bar{x}$ over A such that if $\varphi(\bar{x})$ is such a formula then $\varphi(\bar{a})$ is true in Γ. If $A = \emptyset$ we will simply write $\mathrm{tp}(\bar{a})$ rather than the full $\mathrm{tp}(\bar{a}/\emptyset)$.

Example. If $\Gamma = \mathbb{Z}$ and $x \in \mathbb{Z}[r]$ is arbitrary (see definition 2.2.3) then the true basic formulas over $\emptyset$ are

$$\gamma_0 + \gamma_1 x > 0 \quad \text{where either } \gamma_0 > 0 \text{ and } \gamma_1 = 0, \text{ or } \gamma_1 > 0,$$
$$\text{and} \quad x \equiv r_n \pmod{n} \quad \text{where } n \in \mathbb{N}.$$

As can be seen, the set of these true formulas corresponds to the type of x over $\emptyset$; $\mathrm{tp}(x/\emptyset)$ or simply $\mathrm{tp}(x)$.

Definition 5.2.3. We say that $r(\bar{x}) \supseteq p(\bar{x})$ is **consistent** if $r(\bar{x})$ is a set of basic formulas over Γ_1 and for all finite $q(\bar{x}) \subseteq r(\bar{x})$ there is some $\bar{b} \in \Gamma_1$ with $q(\bar{b}) \subseteq \mathrm{tp}(\bar{b}/\Gamma_1)$.

The following is a version of the compactness theorem, phrased using the notation introduced in this chapter. Although stated for the theory of Presburger arithmetic, it could be used in this form to apply to any complete theory.

Theorem 5.2.4. Let Γ_1 be a Presburger group and $p(\bar{x})$ a set of basic formulas over Γ_1. Then the following are equivalent :-

1. There is some Presburger group $\Gamma_2 \supseteq \Gamma_1$ with $\bar{a} \in \Gamma_2$ such that $p(\bar{a}) \subseteq \mathrm{tp}(\bar{a}/\Gamma_1)$;

2. For any finite $q(\bar{x}) \subseteq p(\bar{x})$ there is $\bar{b} \in \Gamma_1$ such that $q(\bar{b}) \subseteq \mathrm{tp}(\bar{b}/\Gamma_1)$.

Proof. (1) $\implies$ (2) Given $\bar{a} \in \Gamma_2 \supseteq \Gamma_1$ with $p(\bar{a}) \subseteq \text{tp}(\bar{a}/\Gamma_1)$ and finite $q(\bar{x}) \subseteq p(\bar{x})$, let

$$\begin{aligned} C &= \left\{ \zeta : \zeta + \sum_i \gamma_i x_{j_i} > 0 \text{ occurs in } q(\bar{x}) \right\} \\ &= \bar{c}, \text{ a finite set.} \end{aligned}$$

Also, let $n \in \mathbb{N}$ be such that

$$n > \max \{ m : x_i \equiv r \pmod{m} \text{ occurs in } q(\bar{x}) \}$$

and

$$n > \max \left\{ |\gamma_i| : \zeta + \sum_i \gamma_i x_{j_i} > 0 \text{ occurs in } q(\bar{x}) \right\}.$$

Obviously

$$(\Gamma_2, \bar{c}) \underset{0,n'}{\smile} (\Gamma_1, \bar{c}) \quad \text{for all } n',$$

hence by 4.2.1 (elimination of quantifiers)

$$(\Gamma_2, \bar{c}) \underset{j,n}{\smile} (\Gamma_1, \bar{c}) \quad \text{where } j = \text{len}(\bar{a}).$$

Therefore by definition of $\underset{j,n}{\smile}$ there are $\bar{b} \in \Gamma_1$ such that

$$(\Gamma_2, \bar{c}, \bar{a}) \underset{0,n}{\smile} (\Gamma_1, \bar{c}, \bar{b}).$$

i.e. $\text{tp}(\bar{b}/\bar{c}) \supseteq q(\bar{b})$ as required.

(2) $\implies$ (1) By a Zorn's lemma argument we can see that there is a maximal consistent $r(\bar{x}) \supseteq p(\bar{x})$.

For each basic formula

$$\theta(\bar{x}) = \zeta + \sum_i \gamma_i x_{j_i} > 0,$$

either it or its negation

$$\neg\theta(\bar{x}) = (-\zeta + 1) + \sum_i (-\gamma_i) x_{j_i} > 0$$

must be contained in $r(\bar{x})$. If this were not the case, there would be finite sets $q_1(\bar{x}) \subseteq r(\bar{x})$ and $q_2(\bar{x}) \subseteq r(\bar{x})$ for which neither $q_1(\bar{x}) \cup \{\theta(\bar{x})\}$ nor $q_2(\bar{x}) \cup \{\neg\theta(\bar{x})\}$ are satisfied.

But for each $\bar{b} \in \Gamma_1$ one of $\theta(\bar{x})$ or $\neg\theta(\bar{x})$ must be satisfied. Hence $q_1(\bar{x}) \cup q_2(\bar{x})$ is not satisfied, which contradicts the fact that $r(\bar{x})$ is.

A similar argument will show that for all $x_i \in \{\bar{x}\}$ and all $n \in \mathbb{N}$ there is $0 \leq r < n$ with

$$`x_i \equiv r \pmod{n}' \in r(\bar{x}).$$

We now define $\Gamma_2 \supseteq \Gamma_1$ to be the set of all elements of the form

$$\zeta + \sum_i \gamma_i \left(\frac{X_{j_i} - r_i}{m} \right)$$

where $\zeta \in \Gamma_1, `X_{j_i} \equiv r_i \pmod{m}' \in r(\bar{x}), \bar{\gamma} \in \mathbb{Z}$ and $m \in \mathbb{N}$ and where we include elements modulo the standard equivalence relation (cf. definition 2.2.3).

Note that if $`X_i \equiv r \pmod{nm}' \in r(\bar{x})$ then $`X_i \equiv s \pmod{n}' \in r(\bar{x})$ where $r \equiv s \pmod{n}$. We see that this is also true for every $b \in \Gamma_1$, and so, referring once again to definition 2.2.3, we may define addition in Γ_2 so that it becomes an abelian group. Furthermore, we can set

$$\zeta + \sum_i \gamma_i \left(\frac{X_{j_i} - r_i}{m} \right) > 0$$

precisely when

$$`\left(m\zeta - \sum_i r_i \gamma_i \right) + \sum_i \gamma_i X_{j_i} > 0' \in r(\bar{x}),$$

which defines a positive cone and hence can be shown to extend to a complete order relation on Γ_2. This order relation can also be shown to be a discrete linear order respecting + by using a similar argument to that given in the proof of theorem 3.2.1.

We then clearly have $\bar{X} \in \Gamma_2$ such that $p(\bar{X}) \subseteq \mathrm{tp}(\bar{X}/\Gamma_1)$ as required.

□

Definition 5.2.5. Let $A \subseteq \Gamma$, where Γ is a Presburger group. A set of basic formulas in $\bar{x} = (x_0, \ldots, x_j, \ldots)_{j<\alpha}$ over A is called a **semi-type** if it is consistent (*i.e.* every finite subset is realised in Γ). It is a **type** if it is maximally consistent.

Definition 5.2.6. We define $\mathrm{S}_n(A)$ to be the set of all types in $\bar{x} = (x_0, \ldots, x_j, \ldots)_{j<n}$ over A.

It should be noted that our definition of types is somewhat non-standard, as it is restricted to the basic formulas, rather than all formulas of the language. From the

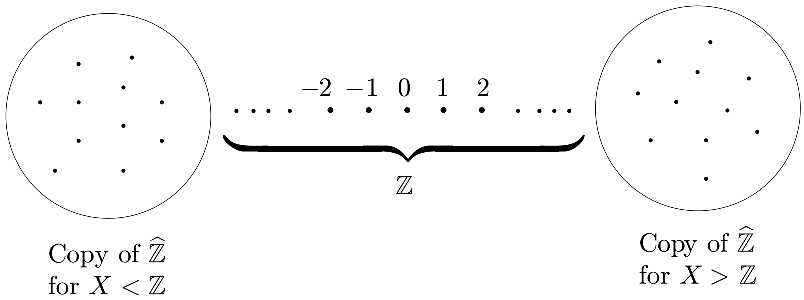

Figure 5.1: Representation of $\mathrm{S}_1(\emptyset)$.

quantifier elimination result of the last chapter we discover that the set of deductive consequences of the formulas in a type as described here constitute a type in the usual model-theoretic sense. Quantifier elimination also tells us that there is therefore a bijective correspondence between the two notions.

Example. If we consider the set $\mathrm{S}_1(\emptyset)$, we see that the basic formulas in X are:

$$z + wX > 0 \quad (z, w \in \mathbb{Z}),$$
$$X \equiv r \pmod{n} \quad (0 \leq r < n \in \mathbb{N}).$$

Consistent sets of formulas can say

$$X = z \tag{5.1}$$

or

$$X > n \quad \text{for all } n \in \mathbb{N}, X \equiv r_i \pmod{i} \tag{5.2}$$

or

$$X < -n \quad \text{for all } n \in \mathbb{N}, X \equiv r_i \pmod{i}, \tag{5.3}$$

where the formula 5.1 is given by $\{(-z+1) + X > 0, (z+1) - X > 0\}$ which has only one extension to a type.

The set $\mathrm{S}_1(\emptyset)$ can then be represented by the diagram shown in Fig. 5.1. The copy of $\mathbb{Z}$ in the central section corresponds to the formulas of type 5.1 with each point (labelled as some $n \in \mathbb{Z}$) representing a type $\mathrm{tp}(n/\emptyset)$. The right hand circle corresponds to elements given by formulas of type 5.2. Every element satisfying such a type will be larger than every element of $\mathbb{Z}$ and hence larger than $\mathbb{Z}$. However since the only constants available for use in formulas are elements of $\mathbb{Z}$ it is not possible to apply the order relation between pairs of elements within this circle, which the depiction of figure 5.1 is intended to suggest. What we do know about the elements in this right hand circle is their residues. It is clear that this circle therefore represents a copy of $\widehat{\mathbb{Z}}$, since every possible residue can be expressed by a set of formulas of the form given in 5.2. These comments apply equally to the formulas of type 5.3, apart from the obvious difference that every element satisfying such a formula will be less than $\mathbb{Z}$ according to the order relation $<$.

5.3 The topology of $\mathrm{S}_n(A)$

As we have seen previously,

$$\begin{aligned}\widehat{\mathbb{Z}} &\subseteq \{\, (r_1, r_2, r_3, \dots) : 0 \leq r_i < i \text{for all } i \,\} \\ &= \prod_{i \in \mathbb{N}} \mathbb{Z}_i,\end{aligned}$$

and so $\widehat{\mathbb{Z}}$ has a natural topology. This natural topology is the subspace topology of $\prod_i \mathbb{Z}_i$ where each $\mathbb{Z}_i$ takes the discrete topology with $\prod_i \mathbb{Z}_i$ having the product topology derived from these.

In effect, the basic open sets of this topology are the finite intersections of those open sets which take the form $\mathcal{U}_{i,k} = \{\langle \bar{r} \rangle : r_i = k\}$. A notable feature of this topology is that the projective maps $\pi_i \colon \prod_i \mathbb{Z}_i \to \mathbb{Z}$ are all continuous.

We shall generalise this topology to a topology for $\mathrm{S}_n(A)$.

Definition 5.3.1. For a basic formula $\theta(\bar{x})$ over A we define

$$\mathcal{U}_\theta = \{\, p(\bar{x}) \in \mathrm{S}_n(A) : \theta(\bar{x}) \in p(\bar{x}) \,\}\,.$$

For example, if θ is the formula '$X_0 \equiv r \pmod{n}$', $\mathcal{U}_\theta$ is the *set of types extending* this.

$$\begin{aligned}\mathcal{U}_f &= \bigcap_{\theta \in f} \mathcal{U}_\theta \qquad \text{for } \textit{finite} \text{ semitypes } f(\bar{x}), \\ &= \{\, p(\bar{x}) \in \mathrm{S}_n(A) : \theta(\bar{x}) \in p(\bar{x}) \text{ for all } \theta \in f \,\}\,.\end{aligned}$$

$\mathrm{S}_n(A)$ can then be given the topology with open sets being the unions of sets of the form $\mathcal{U}_f$. In other words, the union of sets of types which extend some $f(\bar{x}) = \{\theta_i(\bar{x}), \dots, \theta_n(\bar{x})\}$.

Example. Referring once again to fig.5.1 we can consider this topology as it applies to $S_1(\emptyset)$. The copy if $\mathbb{Z}$ in the centre is comprised of isolated points, all of which are open. Hence as a union of open sets this copy of $\mathbb{Z}$ is also open. The two copies of $\widehat{\mathbb{Z}}$ are therefore both closed but not clopen. Each individual copy has the topology of $\widehat{\mathbb{Z}}$ as described above. To show that the copies of $\widehat{\mathbb{Z}}$ are not open suppose $p(\bar{x}) \in \widehat{\mathbb{Z}}$. Then $p(\bar{x}) \in \mathcal{U}_{f(x)}$ where $f(x)$ consists of finitely many basic formulas. However, since there are only finitely many of these formulas in $f(x)$, they will be finitely satisfied in $\mathbb{Z}$. So some $n \in \mathbb{N}$ satisfies $f(x)$ and it follows that one of the isolated types '$x = n$' is in $\mathcal{U}_{f(x)}$. Every open set therefore must contain one of the isolated points from $\mathbb{Z}$.

The implication (2) $\implies$ (3) of the following proposition can be seen to be a refinement of the omitting types theorem for the theory of Presburger arithmetic (for general omitting types see for example Keisler and Chang [33, pp. 79–92]). In the present form we find that the model $\Gamma = \mathrm{cl}(A)$ realises precisely those types which would usually be referred to as *principal types*, or in the present case those types which contain $\{ X_i = \gamma_i : 0 \leq i < n \}$ for some $\bar{\gamma} \in \Gamma$.

To simplify the proof we have taken *isolated types* to mean types satisfied by only a single point. Although this is not the standard usage of the term, in the case of Presburger arithmetic the meaning is equivalent; a fact which follows from quantifier elimination.

Proposition 5.3.2. Let $A \subseteq \Gamma = \mathrm{cl}(A)$ and $p(\bar{x}) \in S_n(A)$.

Then the following are equivalent :

1. $p(\bar{x})$ is isolated;
2. $p(\bar{x})$ is realised in Γ;
3. $p(\bar{x})$ contains $\{ X_i = \gamma_i : 0 \leq i < n \}$ for some $\bar{\gamma} \in \Gamma$.

Proof. (1) $\implies$ (2) Clearly if $p(\bar{x})$ is realised by a single point then it is at any rate realised, and hence (2) follows immediately from (1).

(2) $\implies$ (3) Supposing (2) to be the case we see that $p(\bar{x})$ is realised by some element $\bar{\gamma} \in \Gamma$, say. It may also be realised by other elements of Γ. But $\Gamma = \mathrm{cl}(A)$ and so $\bar{\gamma}$ is definable over A. Now by definition of $S_n(A), p(\bar{x})$ must be maximal, and so for each $0 \leq i < n$ we must have either the formula '$x_i = \gamma_i$' or its negation in $p(\bar{x})$. Clearly the negations cannot be in $p(\bar{x})$ since then $\bar{\gamma}$ would not satisfy it. We therefore conclude that the statement in (3) must be the case.

(3) $\implies$ (1) Assuming the statement in (3) it is clear that if $p(\bar{x})$ is satisfied by anything it can only be satisfied by $\bar{\gamma}$. But $p(\bar{x})$ is a type and so must be consistent, hence is indeed satisfied by something. Thus clearly it is satisfied by the point $\bar{\gamma} \in \Gamma$ and no others. $\square$

Since Γ in the above proposition is also the smallest model containing A and will be embeddable in any larger model also containing A it is clear that should the type $p(\bar{x})$ be as in (3) above, it will in fact be realised in every model $\Gamma \supseteq A$. This constitutes the converse of the omitting types theorem which, unlike the omitting types theorem, only holds in general for complete theories such as we have in this case.

Proposition 5.3.3. For countable $A \subseteq \Gamma = \mathrm{cl}(A)$ and $\alpha \in \mathbb{N}$, the topological space $\mathrm{S}_\alpha(A)$ is complete-metrizable and separable (*i.e.* Polish).

Proof. Enumerate all basic formulas as $(\theta_i(\bar{x}))_{i\in\mathbb{N}}$. For $p \neq q$ in $\mathrm{S}_\alpha(A)$ define $d(p,q) = \frac{1}{i}$ for $i = \min\{\, i : \theta_i \in p \not\Longleftrightarrow \theta_i \in q \,\} = \min\{\, i : (\theta_i \in p \wedge \theta_i \notin q) \vee (\theta_i \notin p \wedge \theta_i \in q)\}$.

If (p_n) is a Cauchy sequence, define

$$\begin{aligned} p &= \{\, \theta : \theta \in p_n \text{ for all but finitely many n} \,\} \\ &= \lim_{n\to\infty} p_n \quad \in \mathrm{S}_\alpha(A). \end{aligned}$$

$\mathrm{S}_\alpha(A)$ is obviously separable, since it is countable. $\square$

Theorem 5.3.4 (Baire). Let X be Polish (cf. 5.3.3). If $U = \bigcap_{i\in\mathbb{N}} U_i$ is a countable intersection of dense open subsets $U_i \subseteq X$ then U is dense in X.

Theorem 5.3.5. There is $r \in \widehat{\mathbb{Z}}$ such that $\mathbb{Z}[r] \xrightarrow{\varrho_r} \widehat{\mathbb{Z}}$ is an injective residue map.

Proof. For each $\lambda, \mu \in \mathbb{Z}$ and $m > 0$ in $\mathbb{Z}$ with $\lambda \neq 0$ let

$$U_i = U_{\langle\lambda,\mu,m\rangle} = \left\{ r \in \widehat{\mathbb{Z}} : r = (\dots, r_m, \dots) \text{ and } \lambda\left(\frac{X - r_m}{m}\right) + \mu \notin \ker \varrho_r \right\}.$$

It suffices to show that each U_i is open and dense. We prove each of these separately.

To show that U_i is open, we take $r = (\dots, r_m, \dots) \in U_i = U_{\langle\lambda,\mu,m\rangle}$. We can find $k \in \mathbb{N}$ such that

$$\begin{aligned} \lambda\left(\frac{X - r_m}{m}\right) + \mu &= \lambda k\left(\frac{X - r_{km}}{km}\right) + \lambda\left(\frac{r_{km} - r_m}{m}\right) + \mu, \\ &\equiv \frac{\lambda(r_{km} - r_m)}{m} + \mu \not\equiv 0 \pmod{k}. \end{aligned}$$

So by virtue of the cross product of discrete topologies being considered, we see that the open set of all

$$r' = (\dots, r_m, \dots, r_{km}, \dots)$$

contains r and is contained in U_i. Hence U_i is indeed open.

To show that U_i is dense, let $n, t \in \mathbb{N}$ with $m \leq n$. It suffices to show that $U_{\langle \lambda, \mu, m \rangle} \cap U \neq \emptyset$ where

$$U = \{ r = (\dots, r_i, \dots) \in \widehat{\mathbb{Z}} : r_i \equiv t \pmod{i} \text{ for } i = 1, 2, \dots, n \}.$$

U represents the inverse image of t using continuous projections to $\mathbb{Z}_i$. Now Let $l = \operatorname{lcm}(1, 2, \dots, n)$ and choose a prime p with $p > l, |\lambda|$. Put $0 \leq r_m < m$ with $t \equiv r_m \pmod{m}$. Also, let

$$U' = \{ r = (\dots, r_i, \dots) : 0 \leq r_{lp} = s < lp \}.$$

We wish to find some $s \equiv t \pmod{l}$ and some congruence class of $s \pmod{lp}$ so that

$$\left(\lambda \left(\frac{X - r_m}{m} \right) + \mu \right) \overset{\varrho_r}{\mapsto} \widehat{\mathbb{Z}} \to \mathbb{Z}_p$$

does not map to zero. The intention will be for us to take $s \equiv r_{lp} \pmod{lp}$. In this case we will have $\emptyset \neq U' \subseteq U$ and the proof will be complete.

So using Bézout's theorem we find $0 \leq y < p$ with

$$-y \equiv \left(\frac{l}{m} \right)^{-1} \left(\left(\frac{t - r_m}{m} \right) + (\mu - 1)\lambda^{-1} \right) \pmod{p}.$$

We note that both inverses exist in $\mathbb{Z}_p$ since $p \nmid l$ whilst $m \mid l$ and $0 < |\lambda| < p$.

Putting $s = t + ly$ we have, for any $r \in U'$, in $\mathbb{Z}[r]$, that

$$\begin{aligned} \lambda \left(\frac{X - r_m}{m} \right) + \mu &= \lambda \frac{lp}{m} \left(\frac{X - s}{lp} \right) + \lambda \left(\frac{s - r_m}{m} \right) + \mu, \\ &\equiv \lambda \left(\frac{s - r_m}{m} \right) + \mu \pmod{p}, \end{aligned}$$

and

$$\begin{aligned} \lambda \left(\frac{s - r_m}{m} \right) + \mu &= \lambda \left(\frac{t + ly - r_m}{m} \right) + \mu, \\ &= \lambda \left(\frac{t - r_m}{m} \right) + \lambda y \left(\frac{l}{m} \right) + \mu, \\ &\equiv \lambda \left(\frac{t - r_m}{m} \right) - \lambda \left(\left(\frac{t - r_m}{m} \right) + (\mu - 1)\lambda^{-1} \right) + \mu \pmod{p}, \\ &\equiv 1 \pmod{p}. \end{aligned}$$

Hence we see that

$$\lambda\left(\frac{X - r_m}{m}\right) + \mu \notin \ker \varrho_r.$$

Therefore $r = (\dots, r_m, \dots) \in U_{\langle\lambda,\mu,m\rangle}$ and so $U_{\langle\lambda,\mu,m\rangle} \cap U \neq \emptyset$ as required. □

The previous theorem (5.3.5) will be improved in the next chapter. This more generalised form, which can be found as lemma 6.2.4, is used to exhibit the fact that $\widehat{\mathbb{Z}}$ may be considered as a Presburger group when furnished with a suitable ordering.

Chapter 6

Compactness

6.1 Pseudotypes

In this section we introduce the notion of pseudotypes. Pseudotypes are not particularly interesting in themselves, however the types which were discussed in the previous chapter form a subset of the pseudotypes. Our sole motivation for introducing them, therefore, is to simplify the proof in the next section of theorem 6.2.2 which concerns the much more useful space of types.

Throughout this section we take A to be a subset of some arbitrary Presburger group Γ.

Definition 6.1.1. A set of basic formulas $p(\bar{x})$ over A is said to be **simply consistent** if $p(\bar{x})$ does not contain

either both of $\sum \lambda_i X_i + \mu > 0$;

$\sum(-\lambda_i)X_i + (1-\mu) > 0 \quad$ for $\mu \in \mathrm{cl}(A)$;

or both of $X_i \equiv r \pmod{n}$;

$X_i \equiv s \pmod{n} \quad$ for $r \neq s$.

Definition 6.1.2. A **pseudotype** is a maximal, simply consistent set of basic formulas. We call the space of pseudotypes $\mathrm{PS}_n(A)$.

The space of pseudotypes $\mathrm{PS}_n(A)$ can be illustrated in the following way:

$$\mathrm{PS}_n(A) \cong \underbrace{\prod_{0\leq i<n} \Big(\prod_{j\in\mathbb{N}} \mathbb{Z}_j \Big)}_{\text{congruence of } X_i} \times \underbrace{\prod_{\substack{\bar\lambda\in\mathbb{Z}^n \\ \mu\in\mathrm{cl}(A)}} \{0,1\}_{\bar\lambda,\mu}}_{\substack{\text{order relations} \\ 1=\text{true} \\ 0=\text{false}}} .$$

As a rough explanation of this representation, suppose we were to construct an element $p(\bar{x})$ of $\mathrm{PS}_n(A)$. To do this we concentrate first on the congruences and see that for each X_i where $0 \leq i < n$ we must select a maximal simply consistent set of congruence formulas. This involves selecting an element $r_j \in \mathbb{Z}_j$ for each $j \in \mathbb{N}$ in order that '$X_i \equiv r_j \pmod{j}$' $\in p(\bar{x})$. To ensure that this is maximal we must choose at least one such element r_j for each $j \in \mathbb{N}$ and to remain simply consistent we can choose no more than one such element. The possible congruences will therefore be isomorphic to the set $\prod_{0\leq i<n} \prod_{j\in\mathbb{N}} \mathbb{Z}_j$. Having chosen congruences we are then free to choose the order relations. The fact that the possible order relations are isomorphic to the set $\prod_{\bar\lambda\in\mathbb{Z}^n,\mu\in\mathrm{cl}(A)}\{0,1\}_{\bar\lambda,\mu}$ can be explained in a similar to that just given for congruences. We conclude that $\mathrm{PS}_n(A)$ can indeed be represented in the manner given above.

6.2 Compactness

Definition 6.2.1. Suppose Y is a topological space. We say that $x, y \in Y$ are **disconnected** if there exist open $U, V \subseteq Y$ with $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$ and $U \cup V = Y$. We say that Y is **totally disconnected** if every pair of distinct points $x, y \in Y$ is disconnected.

Theorem 6.2.2. $\mathrm{S}_n(A)$ is compact and totally disconnected.

Proof. We first note that $\mathrm{PS}_n(A)$ is compact by Tychonoff. But now $\mathrm{S}_n(A) \subseteq \mathrm{PS}_n(A)$ and if $p \in \mathrm{PS}_n(A) \setminus \mathrm{S}_n(A)$ then p is simply consistent but not consistent. Hence there is a finite subset — the basic open neighbourhood of p — which is not realised in $\mathrm{cl}(A)$. So $\mathrm{S}_n(A)$ is seen to be a closed subset of a compact space, and is therefore also compact.

To show that $\mathrm{S}_n(A)$ is totally disconnected, take $p, q \in \mathrm{S}_n(A)$ with $p \neq q$. Then either

1. p contains $\underbrace{\sum \lambda_i X_i + \mu > 0}_{\theta}$ and q contains $\underbrace{\sum (-\lambda_i) X_i + (1-\mu) > 0}_{\neg\theta}$ or

2. p contains $\underbrace{X_i \equiv j \pmod{n}}_{\theta_j}$ and q contains $\underbrace{X_i \equiv k \pmod{n}}_{\theta_k}$

In the first case we have $\mathrm{S}_n(A) = U_\theta \dot\cup U_{\neg\theta}$ and in the second we have $\mathrm{S}_n(A) = U_{\theta_0} \dot\cup \ldots \dot\cup U_{\theta_{n-1}}$. Either way, we see that $\mathrm{S}_n(A)$ is disconnected. □

The compactness of $\mathrm{S}_n(A)$ has been established here using topological techniques and has been described as a topological property. This topological compactness is however equivalent to a restricted version of what might ordinarily be described as logical compactness and which states that a set of sentences Σ in a language $\mathscr{L}$ is consistent (*i.e.* satisfiable) if and only if it is finitely satisfiable. The topological compactness of $\mathrm{S}_n(\emptyset)$ is equivalent to the statement in which we are considering only the language $\mathscr{L} = \{+, <, 0, 1, x_1, \ldots, x_n\}$, with $x_1, \ldots, x_n$ being arbitrary constant symbols, and in which all sets of sentences Σ are extensions of the standard Presburger axioms. More generally, the compactness of $\mathrm{S}_n(A)$ is equivalent to the case for $\mathrm{S}_n(\emptyset)$ with the additional requirements that our language is also extended with a constant symbol for each element $a_i \in A$, and in which Σ must also contain all true sentences constituting properties of these a_i in a model of Presburger arithmetic Γ. Given these restrictions, the two notions of compactness are precisely equivalent. If we denote the set of Presburger axioms as $\mathbf{Pr}$, then we may paraphrase the above in the following manner:

Proposition 6.2.3. Suppose Γ is a Presburger group, $A \subseteq \Gamma$ and let Σ be a set of sentences in the language

$$\mathscr{L} = \{+, <, 0, 1, x_1, \ldots, x_n\} \cup \{\, a_i : a_i \in A\}$$

which extends

$$\mathbf{Pr} \cup \{\, \theta(a_1, \ldots a_m) : a_1, \ldots, a_m \in A \text{ and } \Gamma \models \theta(a_1, \ldots a_m)\}.$$

Then $\Gamma \models \Sigma \iff \Gamma \models \Sigma'$ for every finite subset $\Sigma' \subseteq \Sigma$.

The following lemma is an extension of theorem 5.3.5 given at the conclusion of the previous chapter.

Theorem 6.2.4. Let $\Gamma \leq \widehat{\mathbb{Z}}$ be such that there is a discrete linear order $<$ on Γ making Γ into a Presburger group with least positive element $\widehat{1} = (0, 1, 1, \dots)$. Suppose also that $r \in \widehat{\mathbb{Z}} \setminus \Gamma$. Then the map

$$\Gamma[r] \xrightarrow{\varrho_r} \widehat{\mathbb{Z}}$$

is an injective residue map.

Proof. A typical element of $\Gamma[r]$ is of the form

$$t = \lambda \frac{X - r_n}{n} + \mu$$

where $n \in \mathbb{N}, \lambda \in \mathbb{Z}$ and $\mu \in \Gamma$. We first show that $\varrho_r(t) \neq 0$ in the case where $\lambda = 1$. Let $m \in \mathbb{N}$ and $p \in \mathbb{N}$ be prime and suppose $\varrho_r(t) = 0$.

$$\begin{aligned} t &= \frac{X - r_n}{n} + \mu = p^m \frac{X - r_{n \cdot p^m}}{n \cdot p^m} + \frac{r_{n \cdot p^m m} - r_n}{n} + \mu, \\ &\equiv \frac{r_{n \cdot p^m} - r_n}{n} + \mu \pmod{p^m}. \end{aligned}$$

Also $r_{n \cdot p^m} = r_n + A_{p^m} \cdot n$ where $0 \leq A_{p^m} < p^m$. So

$$\begin{aligned} t &\equiv A_{p^m} + \mu, \\ &\equiv 0 \pmod{p^m}, \\ A_{p^m} &\equiv -\mu \pmod{p^m}, \end{aligned}$$

and

$$\begin{aligned} r_{n \cdot p^m} &= r_n + A_{p^m} \cdot n, \\ &\equiv r_n - \mu n \pmod{p^m}. \end{aligned}$$

These equations determine r_k for all k :-

$$r_{p^m} \equiv r_n - \mu n \pmod{p^m}.$$

But $\mu \in \Gamma, r_n \in \mathbb{Z} \subseteq \Gamma$ so $r_n - \mu n \in \Gamma$ and hence $r \in \Gamma$, a contradiction.

We now do the case when $\lambda \neq 0$ is arbitrary. So we consider

$$\varrho_r \colon t = \lambda \frac{X - r_n}{n} + \mu \mapsto 0 \in \widehat{\mathbb{Z}}.$$

Note that $\lambda \in \mathbb{Z}$ and we may assume $\lambda \geq 2$.

Also

$$\lambda\frac{X-r_n}{n}+\mu\equiv\mu \pmod{\lambda}$$

and since this is the image of t under

$$\mathbb{Z}[r]\xrightarrow{\varrho_r}\widehat{\mathbb{Z}}\to\mathbb{Z}_\lambda$$

we must have $\mu=\lambda\nu$ for some $\nu\in\Gamma$, since $\varrho_r(t)=0$.

Hence $t=\lambda\left(\frac{X-r_n}{n}+\nu\right)$ and

$$\varrho_r\left(\frac{X-r_n}{n}+\nu\right)\in\widehat{\mathbb{Z}}.$$

But $\lambda\varrho_r\left(\frac{X-r_n}{n}+\nu\right)=0$ and therefore by Lagrange $\varrho_r\left(\frac{X-r_n}{n}+\nu\right)$ must have order dividing λ.

By the previous part, $\varrho_r\left(\frac{X-r_n}{n}+\nu\right)\neq 0$.

Choose $p\,|\,\lambda$ a prime and $\lambda=p^m q$ with p not dividing q. Define z_{p^m} to be the image of $\frac{X-r_n}{n}+\nu$ under $\Gamma[r]\xrightarrow{\varrho_r}\widehat{\mathbb{Z}}\to\mathbb{Z}_{p^m}$. Then z_{p^m} has order p^k in $\mathbb{Z}_{p^m}$ for $0\leq k\leq m$. Also, for some prime $p\,|\,\lambda$ we see that z_{p^m} has non-trivial order, since $\varrho_r\left(\frac{X-r_n}{n}+\nu\right)\neq 0$, so without loss of generality $0<k\leq m$.

Therefore, in $\mathbb{Z}_{p^m}$, $z_{p^m}=Ap^{m-k}$ for some A not divisible by p. In $\mathbb{Z}_{p^{2m-k+1}}$ we have $z_{p^{2m-k+1}}=Bp^m+Ap^{m-k}$ for some B with $0\leq B<p^{m-k+1}$.

Therefore $\langle z_{p^{2m-k+1}}\rangle$ is cyclic of order p^{m+1} not dividing λ. This is impossible, and so once again we have a contradiction. □

The forthcoming result is an obvious corollary of theorem 6.2.4 above. It is particularly interesting since it provides a tangible example of a nonstandard model of Presburger arithmetic in the form of the group $\widehat{\mathbb{Z}}$ with a discrete linear order. Its usefulness is complemented by the fact that $\widehat{\mathbb{Z}}$ is a well understood structure and hence provides a good starting point when considering the applicability of results to Presburger arithmetic in general.

Both theorem 6.2.4 and its corollary below are results of Richard Kaye and can be found in [31].

Corollary 6.2.5. There is a discrete linear order on $\widehat{\mathbb{Z}}$ with least positive element $\widehat{1}$ making $\widehat{\mathbb{Z}}$ a Presburger group.

Proof. Apply a Zorn's lemma argument to the previous theorem. □

Chapter 7

Homogeneous Presburger Groups

7.1 Homogeneity

When considering a Presburger group Γ we may ask the question of what conditions are necessary in order that for two points $\bar{x}, \bar{y} \in \Gamma^n$ we may find an automorphism $\alpha \in \text{Aut}(\Gamma)$ such that $\bar{x}\alpha = \bar{y}$. Clearly such a necessary condition will be that $\text{tp}(\bar{x}) = \text{tp}(\bar{y})$, but is this sufficient? It turns out that it is not, as the next example illustrates:

Example. Suppose $r \in \widehat{\mathbb{Z}} \setminus \mathbb{Z}$. With reference to definition 2.2.4 we may extend $\mathbb{Z}$ upwards with distinct elements x, y both having residue r to obtain the Presburger group $\mathbb{Z}[x, y]$ (see Fig. 7.1 where $\Gamma = \text{cl}(x, y)$). From the construction we know that $\mathbb{Z} < \text{cl}(x) < y$. We also know that $\text{tp}(x) = \text{tp}(y)$, however we can easily show that there is no order automorphism $\alpha \in \text{Aut}(\Gamma)$ which maps x to y.

For suppose there were such an automorphism. Then we see that for all $\gamma \in \Gamma$ there is $k \in \mathbb{N}$ such that $ky > \gamma$. But $y\alpha \in \Gamma$ so for some $k \in \mathbb{N}$ we have that $ky > y\alpha$. However for all $k \in \mathbb{N}$ it is clear that $kx < y$ so $kx\alpha < y\alpha$ from which we find that $ky < y\alpha$; a contradiction. There can therefore be no such order automorphism α.

Definition 7.1.1. Γ is **homogeneous** if whenever $n \in \mathbb{N}$ and $\bar{a}, \bar{b} \in \Gamma^n$ with $\text{tp}(\bar{a}) =$

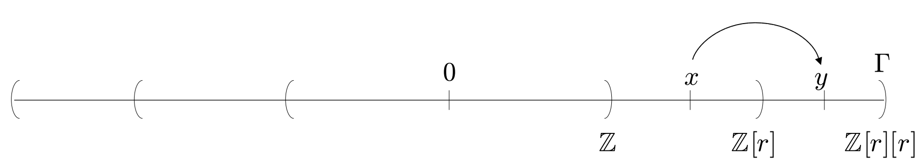

Figure 7.1: Example of a non-homogeneous Presburger group.

$\text{tp}(\bar{b})$ there is $\alpha \in \text{Aut}(\Gamma)$ such that $\bar{a}\alpha = \bar{b}$.

We have seen that not all models of Presburger arithmetic are homogeneous. However as yet we do not know whether any homogeneous models of Presburger arithmetic can actually exist. Using the next few lemmas we are able to show in theorem 7.1.5 that such models do exist, and in fact that we can extend any model of Presburger arithmetic to a homogeneous model without having to add too much to our original model.

Notation. We let $\text{Res}(\Gamma)$ refer to the set of residues of the Presburger group Γ, *i.e.* the image of the natural map $\Gamma \to \widehat{\mathbb{Z}}$.

Lemma 7.1.2. If Γ is a Presburger group, $\bar{x}, \bar{y} \in \Gamma, \text{tp}(\bar{x}) = \text{tp}(\bar{y})$ and $z \in \Gamma$, then there exists $w \in \Gamma' \supseteq \Gamma$ such that $\text{tp}(\bar{x}, z) = \text{tp}(\bar{y}, w)$ and $\text{Res}(\Gamma') = \text{Res}(\Gamma)$.

Proof. Let $p(X) = \text{tp}(z, \bar{x})$ and $q(X) = \{\theta(X, \bar{y}) : \theta(X, \bar{x}) \in p(X)\}$. We note that $\theta(X, \bar{x}) \in p(X)$ if and only if $\Gamma \models \theta(z, \bar{x})$.

We want to show that $q(X)$ is finitely satisfiable. So take a finite subset

$$S = \{\theta_1(X, \bar{y}), \theta_2(X, \bar{y}), \dots, \theta_n(X, \bar{y})\} \subset p(X).$$

and let

$$\psi(X_1, \bar{X}_2) = \bigwedge_{1 \le i \le n} \theta_i(X_1, \bar{X}_2) \quad \text{ where } \theta_i(X, \bar{y}) \in S.$$

Clearly $\Gamma \models \theta_i(z, \bar{x})$ for $1 \le i \le n$. Hence

$$\Gamma \models \psi(z, \bar{x}) \quad \Rightarrow \quad \Gamma \models \exists\, x_1 \psi(x_1, \bar{x}).$$

By quantifier elimination, $\exists\, x_1 \psi(x_1, \bar{x})$ is equivalent to some basic formula which will occur in $\text{tp}(\bar{x})$, and since $\text{tp}(\bar{x}) = \text{tp}(\bar{y})$ we therefore have that $\Gamma \models \exists\, x_1 \psi(x_1, \bar{y})$. Thus for some $a \in \Gamma$ we have $\Gamma \models \psi(a, \bar{y})$. It follows that any finite subset S of $q(X)$ is satisfied in Γ.

Now by the implication (2) $\implies$ (1) of theorem 5.2.4, and taking $p(\bar{x})$ to be complete, it follows that we can find some $\Gamma' \supseteq \Gamma$ with $w \in \Gamma'$ such that $q(w) = \text{tp}(w, \bar{y})$. Hence we have $\text{tp}(\bar{x}, z) = \text{tp}(\bar{y}, w)$.

We consider the proof of this theorem in order to show that $\text{Res}(\Gamma') = \text{Res}(\Gamma)$.

To begin, we claim that $\varrho(z) = \varrho(w)$ (*i.e.* z and w have the same residues). But this is clear, for suppose $\Gamma \models z \equiv r_n \pmod{n}$ for some $r_n, n \in \mathbb{N}$. Then '$z \equiv r_n \pmod{n}$' $\in$

$\text{tp}(\bar{x}, z)$ implies that '$w \equiv r_n \pmod{n}$' $\in \text{tp}(\bar{y}, w)$ so $\Gamma' \models w \equiv r_n \pmod{n}$. Since $\text{tp}(\bar{x}, z)$ is complete, z and w must therefore have the same residues.

Now let $\bar{x} = (x_1, x_2, \ldots, x_k), \bar{y} = (y_1, y_2, \ldots, y_k)$.

For any (other) element $\alpha \in \Gamma'$, by reference to the proof of theorem 5.2.4 we see that

$$\alpha = \zeta + \sum_i \gamma_i \left(\frac{X_{j_i} - r_i}{m} \right)$$

where $\zeta \in \Gamma$, '$X_{j_i} \equiv r_i \pmod{m}$' $\in p(\bar{x}), \bar{\gamma} \in \mathbb{Z}$ and $m \in \mathbb{N}$, taking $p(\bar{x}) = \text{tp}(z, \bar{x})$.

$$\begin{array}{rll} \text{Hence} & X_{j_i} \equiv x_{j_i - 1} \pmod{n} \text{ for all } n \in \mathbb{N} & \text{if } j_i \geq 2 \\ \text{or} & X_{j_i} \equiv z \pmod{n} \text{ for all } n \in \mathbb{N} & \text{otherwise.} \end{array}$$

So

$$\varrho \left(\frac{X_{j_i} - r_i}{m} \right) \in \text{Res}(\Gamma)$$

and hence $\varrho(\alpha) \in \text{Res}(\Gamma)$ from which we may conclude that $\text{Res}(\Gamma') \subseteq \text{Res}(\Gamma)$. But $\Gamma \subseteq \Gamma'$ and so $\text{Res}(\Gamma') = \text{Res}(\Gamma)$ as required. □

In the next lemma we take ω to be the smallest infinite ordinal with its standard ordering.

Lemma 7.1.3. There exists a function $\theta \colon \omega \to \omega \times \omega \times \omega$ which is both onto and ∞ to 1.

Proof. We provide an explicit example of such a function. Define θ to map

$$\theta(n) \mapsto (a, b, c)$$

where $\psi(\varphi(n)) = (a, d)$ and $(b, c) = \psi(\varphi(d))$.

The two maps ψ and φ are as follows:

$\varphi \colon \omega \to \omega$ is the ∞ to 1, onto function defined by

$\varphi(x) \mapsto i$ where P_i is the smallest prime factor of x;

$\psi \colon \omega \to \omega \times \omega$ is the 1 to 1, onto function defined by

$\psi(x) \mapsto (a, b)$ where b is the smallest natural number so that $x = \frac{a(a+1)}{2} + b$. □

Lemma 7.1.4. For all countable models of Presburger arithmetic Γ, there is a countable extension $\Gamma' \supseteq \Gamma$ with $\text{Res}(\Gamma) = \text{Res}(\Gamma')$ and such that for all $\bar{x}, \bar{y} \in \Gamma'$ with $\text{tp}(\bar{x}) = \text{tp}(\bar{y})$ and any $z \in \Gamma'$ there is a $w \in \Gamma'$ such that $\text{tp}(\bar{x}, z) = \text{tp}(\bar{y}, w)$.

Proof. We start by letting $\Gamma^{(1)} = \Gamma$ and then use an iterative process, so that at the n-th step we have

$$\Gamma^{(n)} \supseteq \Gamma^{(n-1)} \supseteq \cdots \supseteq \Gamma^{(1)} \tag{7.1}$$

with $\mathrm{Res}(\Gamma^{(i)}) = \mathrm{Res}(\Gamma^{(i-1)})$ for all $1 < i \leq n$.

Now let $((\bar{x}^i_1, \bar{y}^i_1), (\bar{x}^i_2, \bar{y}^i_2), \dots)$ with $\bar{x}^i_j, \bar{y}^i_j \in \Gamma^{(i)}$ for $j \in \mathbb{N}$ be an enumeration of tuples of $\Gamma^{(i)}$ such that $\mathrm{tp}(\bar{x}^i_j) = \mathrm{tp}(\bar{y}^i_j)$ for each $i \leq n$, and let $(\gamma^i_1, \gamma^i_2, \dots)$ be an enumeration of $\Gamma^{(i)}$.

Let $\theta\colon \omega \to \omega \times \omega \times \omega$ be an onto, ∞ to 1 function the existence of which is given by lemma 7.1.3. Now for the $(n+1)$-th step we have $\theta(n+1) = (a, b, c)$ where $a, b, c \in \mathbb{N}$.

If $a > n$ let $\Gamma^{(n+1)} = \Gamma^{(n)}$. Otherwise $a \leq n$ and so $\Gamma^{(a)}$ has been previously defined as part of the sequence (7.1) above. We can therefore consider the elements $(\bar{x}^a_b, \bar{y}^a_b), \gamma^a_c$. If there already exists an element $w \in \Gamma^{(n)}$ such that $\mathrm{tp}(\bar{x}^a_b, \gamma^a_c) = \mathrm{tp}(\bar{y}^a_b, w)$ then we simply let $\Gamma^{(n+1)} = \Gamma^{(n)}$ as before. Otherwise we use lemma 7.1.2 to extend $\Gamma^{(n)}$ as follows:

By our assumptions we have that $\mathrm{tp}(\bar{x}^a_b) = \mathrm{tp}(\bar{y}^a_b)$ and so by lemma 7.1.2 with $z = \gamma^a_c$ we can find $\Gamma^{(n+1)}$ and $w \in \Gamma^{(n+1)}$ such that $\mathrm{tp}(\bar{x}^a_b, \gamma^a_c) = \mathrm{tp}(\bar{y}^a_b, w)$.

We then fix an enumeration

$$\big((\bar{x}^{n+1}_1, \bar{y}^{n+1}_2), (\bar{x}^{n+1}_2, \bar{y}^{n+1}_2), \dots\big)$$

such that $\mathrm{tp}(\bar{x}^{n+1}_j) = \mathrm{tp}(\bar{y}^{n+1}_j)$ for all $j \in \mathbb{N}$, and an enumeration

$$\big(\gamma^{n+1}_1, \gamma^{n+1}_2, \dots\big) \text{ of } \Gamma^{(n+1)}.$$

This completes the iterative step. It is clear from lemma 7.1.2 that $\mathrm{Res}(\Gamma^{(n+1)}) = \mathrm{Res}(\Gamma^{(n)})$.

We now let

$$\Gamma' = \bigcup_{i \in \omega} \Gamma^{(i)}$$

and we claim that this is our required model. By our construction it is clear that $\mathrm{Res}(\Gamma') = \mathrm{Res}(\Gamma)$, that Γ' is countable and that $\Gamma' \supseteq \Gamma$. If we take arbitrary $\bar{x}, \bar{y}, z \in \Gamma'$ such that $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ then $\bar{x}, \bar{y}, z$ are all finite tuples, and hence to be found in $\Gamma^{(n)}$ for some $n \in \mathbb{N}$. But for some $m \geq n$ we have $\theta(m) = (n, b, c)$ with $\bar{x} = \bar{x}^n_b, \bar{y} = \bar{y}^n_b$ and $z = \gamma^n_c$. Hence by our construction there exists some $w \in \Gamma^{(m+1)} \subseteq \Gamma'$ such that $\mathrm{tp}(\bar{x}, z) = \mathrm{tp}(\bar{y}, w)$ as required. □

The following theorem, which is really just a corollary of the lemma 7.1.4 above, provides us with our desired result that homogeneous models exist, and are in fact relatively easy to find. Since we make extensive use of back-and-forth constructions in this thesis, we have given a full account of the construction below, however in future we will tend to leave out the details of the back-and-forth framework.

Theorem 7.1.5. For all countable models of Presburger arithmetic Γ there is a countable homogeneous $\Gamma' \supseteq \Gamma$ and $\mathrm{Res}(\Gamma') = \mathrm{Res}(\Gamma)$.

Proof. By lemma 7.1.4 we can find a countable extension $\Gamma' \supseteq \Gamma$ such that for all $\bar{x}, \bar{y} \in \Gamma'$ with $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ and any $z \in \Gamma'$ there is a $w \in \Gamma'$ such that $\mathrm{tp}(\bar{x}, z) = \mathrm{tp}(\bar{y}, w)$. We claim that this model Γ' is homogeneous.

So suppose $\bar{x}, \bar{y} \in \Gamma'$ are such that $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$. We must show that we can find an automorphism which maps $\bar{x}$ to $\bar{y}$. We do this by constructing such an automorphism iteratively using back-and-forth.

Let $(\gamma_1, \gamma_2, \dots)$ be an enumeration of Γ', and suppose that at the n-th stage of our iteration we have

$$\mathrm{tp}(\bar{x}, \gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_n}) = \mathrm{tp}(\bar{y}, \gamma_{j_1}, \gamma_{j_2}, \dots, \gamma_{j_n})$$

for some $i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n \in \mathbb{N}$.

If n is even, we let i_{n+1} be the smallest such that $\gamma_{i_{n+1}} \notin \{\bar{x}, \gamma_{i_1}, \dots, \gamma_{i_n}\}$. By lemma 7.1.4 we can let $\gamma_{j_{n+1}} \in \Gamma'$ be such that

$$\mathrm{tp}(\bar{x}, \gamma_{i_1}, \dots, \gamma_{i_n}, \gamma_{i_{n+1}}) = \mathrm{tp}(\bar{y}, \gamma_{j_1}, \dots, \gamma_{j_n}, \gamma_{j_{n+1}}) .$$

If n is odd, we let j_{n+1} be the smallest such that $\gamma_{j_{n+1}} \notin \{\bar{y}, \gamma_{j_1}, \dots, \gamma_{j_n}\}$. By lemma 7.1.4 we can let $\gamma_{i_{n+1}} \in \Gamma'$ be such that

$$\mathrm{tp}(\bar{x}, \gamma_{i_1}, \dots, \gamma_{i_n}, \gamma_{i_{n+1}}) = \mathrm{tp}(\bar{y}, \gamma_{j_1}, \dots, \gamma_{j_n}, \gamma_{j_{n+1}}) .$$

We then iterate this process until $\mathrm{tp}(\bar{x}, \Gamma') = \mathrm{tp}(\bar{y}, \Gamma')$ and then define $\alpha \in \mathrm{Aut}(\Gamma')$ to be $\bar{x}\alpha = \bar{y}$ and $\gamma_{i_m}\alpha = \gamma_{j_m}$ for all $m \in \mathbb{N}$.

We therefore see that Γ' is indeed homogeneous as required. $\square$

Chapter 8

Notions of Independence

8.1 Linear independence and standard parts

In this chapter we will examine notions of independence in a model of Presburger arithmetic. We will make extensive use of the fact that $\Gamma/\mathbb{Z}$ is divisible and torsion free, and hence a vector space over $\mathbb{Q}$.

Definition 8.1.1. We say that $B \subseteq \Gamma/\mathbb{Z}$ is **linearly independent** if it is so as a subset of the $\mathbb{Q}$-vector space as described above.

We say $B \subseteq \Gamma$ is **linearly independent** if it contains at most one representative of each coset of $\mathbb{Z}$ and $B/\mathbb{Z}$ is linearly independent as defined above.

Example. Any singleton set $\{\gamma\}$ is clearly linearly independent as long as $\gamma \notin \mathbb{Z}$.

Ideally we would hope that our notion of independence was strong enough so that for any $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ both linearly independent subsets of $\Gamma/\mathbb{Z}$, such that $\varrho/\mathbb{Z}(a_i) = \varrho/\mathbb{Z}(b_i)$ and $a_i < a_j \iff b_i < b_j$ for all $i, j = 1, 2, \ldots, n$, we are able to extend the map $\alpha_0 \colon a_i \mapsto b_i$ to an automorphism of Γ. Unfortunately this is not the case, and we illustrate this by exhibiting a counter example.

Example. Choose some Presburger group Γ with elements $a, b \in \Gamma$ with $\varrho(a) = \varrho(b) = 0$ and $a > \mathrm{cl}(\emptyset), b > \mathrm{cl}(a)$. Such a Presburger group certainly exists as the constructions in section 2.2 show. We then put $c = a+b$. A representation of this is shown in Fig. 8.1. Then $\varrho(c) = \varrho(a) + \varrho(b) = 0$ and the sets $\{a, b\}, \{b, c\}$ are both linearly independent. For the set $\{b, c\}$, this follows since if it were not the case we would have that $n_1 b = n_2 c$ for some $n_1, n_2 \in \mathbb{Z}$. But then $n_1 b = n_2(a + b)$, so $(n_1 - n_2)b = a$, which contradicts the fact that $b > \mathrm{cl}(a)$.

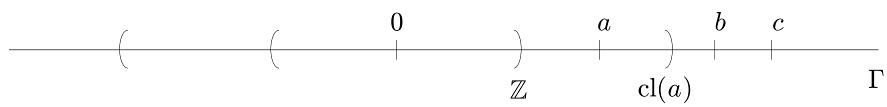

Figure 8.1: Linearly independent subsets not extending to an automorphism.

However, it is clear that there can be no automorphism which takes $a, b \mapsto b, c$, since $b > \mathrm{cl}(a)$ whilst $c \not> \mathrm{cl}(b)$. Figure 8.1 above illustrates such a situation, although despite the appearance of this diagram, it is possible that $\mathrm{cl}(a)$ and $\mathrm{cl}(b)$ may not be convex in Γ.

So in order to be able to extend to automorphisms between certain sets, we need a stronger notion of independence. To establish such a notion we require the following definition:

Definition 8.1.2. For $a, b \in \Gamma$ with $a, b > \mathbb{Z}$, we define

$$\mathrm{st}\left(\frac{a}{b}\right) = \left\{ q \in \mathbb{Q} : \exists\, r, s \in \mathbb{N}, s > 0, \frac{r}{s} > q \text{ and } rb < sa \right\}.$$

This is an extended cut, identified with an extended real $r \in [0, \infty] \subseteq \mathbb{R} \cup \{\infty\}$, where $r = \sup \mathrm{st}\left(\frac{a}{b}\right)$. For some background on such cuts, see Enderton [16, pp. 113–120].

Recalling the definition of $\mathrm{Res}(\Gamma)$ (notation on page 49) it is also useful to have a similar notion for the standard parts.

Notation. Let $\mathrm{stQ}(\Gamma)$ be the set of **standard parts** achievable in Γ;

$$\mathrm{stQ}(\Gamma) = \left\{ \mathrm{st}\left(\frac{a}{b}\right) \in \mathbb{R} \cup \{\pm\infty\} : a, b \in \Gamma \right\}.$$

The following proposition ensures that we can apply this same notation to the coset space $\Gamma/\mathbb{Z}$ in a well defined manner.

Proposition 8.1.3. If $a, b > \mathbb{Z}$ then

$$\mathrm{st}\left(\frac{a+n}{b+m}\right) = \mathrm{st}\left(\frac{a}{b}\right)$$

for all $n, m \in \mathbb{Z}$.

Proof. Fix $n, m \in \mathbb{Z}$. We first show that $\mathrm{st}\left(\frac{a}{b}\right) \subseteq \mathrm{st}\left(\frac{a+n}{b+m}\right)$. For any $q \in \mathrm{st}\left(\frac{a}{b}\right)$ there exists some $r, n \in \mathbb{N}$ as in the definition 8.1.2 so that $rb < sa$ with $r, s \in \mathbb{N}, s > 0$. Let $\frac{r}{s} - q = \varepsilon \in \mathbb{Q}$ and choose some $N \in \mathbb{N}$ with

$$0 < \frac{1}{N} < \varepsilon.$$

Then

$$rNm - sm - sNn \in \mathbb{Z}$$

so we have that

$$\begin{aligned} rNm - sm - sNn <& sb \leq N(sa - rb) + sb, \\ rNm - sm + rNb - sb <& sNa + sNn, \\ (rN - s)(b + m) <& sN(a + n) \end{aligned}$$

and so

$$\frac{r}{s} - \varepsilon < \frac{r}{s} - \frac{1}{N} \in \mathrm{st}\left(\frac{a+n}{b+m}\right).$$

We now show that $\mathrm{st}\left(\frac{a+n}{b+m}\right) \subseteq \mathrm{st}\left(\frac{a}{b}\right)$. Again, for any $q \in \mathrm{st}\left(\frac{a+n}{b+m}\right)$ there exists some $r, n \in \mathbb{N}$ as in the definition 8.1.2 so that $r(b+m) < s(a+n)$ with $r, s \in \mathbb{N}, s > 0$. Let $\frac{r}{s} - q = \varepsilon \in \mathbb{Q}$ and choose some $N \in \mathbb{N}$ with

$$0 < \frac{1}{N} < \varepsilon.$$

Then

$$N(sn - rm) \in \mathbb{Z}$$

and

$$N(sn - rm) < sb,$$

so we have that

$$\begin{aligned} rNb <& sNa + N(sn - rm), \\ rNb <& sNa + sb, \\ (rN - s)b <& sNa \end{aligned}$$

and so

$$\frac{r}{s} - \varepsilon < \frac{r}{s} - \frac{1}{N} \in \mathrm{st}\left(\frac{a}{b}\right).$$

□

Definition 8.1.4. If $\mathbb{Z} + a, \mathbb{Z} + b \in \Gamma/\mathbb{Z}$ and $a, b > \mathbb{Z}$ we set

$$\mathrm{st}\left(\frac{\mathbb{Z}+a}{\mathbb{Z}+b}\right) = \mathrm{st}\left(\frac{a}{b}\right).$$

The previous proposition ensures that this is well-defined. In future results which concern standard parts we will refrain from giving two presentations for both Γ and $\Gamma/\mathbb{Z}$. In all cases, however, a result for one will extend to a result for the other and we will assume this without assertion.

In order to complete the picture where $a, b \in \Gamma$ and $b \neq 0$ we also set:

$$\begin{aligned}
\operatorname{st}\left(\frac{\mathbb{Z}}{\mathbb{Z}+b}\right) &= 0; \\
\operatorname{st}\left(\frac{\mathbb{Z}+a}{\mathbb{Z}}\right) &= \begin{cases} +\infty & a > \mathbb{Z}, \\ 0 & a \in \mathbb{Z}, \\ -\infty & a < \mathbb{Z}; \end{cases} \\
\operatorname{st}\left(\frac{\mathbb{Z}-a}{\mathbb{Z}+b}\right) &= -\operatorname{st}\left(\frac{\mathbb{Z}+a}{\mathbb{Z}+b}\right) = \operatorname{st}\left(\frac{\mathbb{Z}+a}{\mathbb{Z}-b}\right); \\
\operatorname{st}\left(\frac{\mathbb{Z}-a}{\mathbb{Z}-b}\right) &= \operatorname{st}\left(\frac{\mathbb{Z}+a}{\mathbb{Z}+b}\right).
\end{aligned}$$

We may refer to $\operatorname{st}\left(\frac{a}{b}\right)$ as the standard part of $\frac{a}{b}$. It must be remembered, however, that the notation $\frac{a}{b}$ has no real meaning on its own.

For an examination of the properties of standard parts it is useful to recall the operations which are defined on the extended reals $\mathbb{R} \cup \{\pm\infty\}$. In particular we point out that none of $0 \cdot \infty$, $0 \cdot (-\infty)$ or $\infty + (-\infty)$ are defined. With this in mind we note the following useful properties of $\operatorname{st}\left(\frac{a}{b}\right)$ which will be used extensively:

Lemma 8.1.5. For $a, b, c \in \Gamma/\mathbb{Z}, q \in \mathbb{Q}$ the following hold:

1. $\operatorname{st}\left(\frac{a}{b}\right) \cdot \operatorname{st}\left(\frac{b}{c}\right) = \operatorname{st}\left(\frac{a}{c}\right)$ provided the left hand side multiplication is defined;
2. $\operatorname{st}\left(\frac{qa}{b}\right) = q \cdot \operatorname{st}\left(\frac{a}{b}\right)$;
3. $\operatorname{st}\left(\frac{a}{qb}\right) = \frac{1}{q} \cdot \operatorname{st}\left(\frac{a}{b}\right)$ for $q \neq 0$;
4. $\operatorname{st}\left(\frac{a+b}{c}\right) = \operatorname{st}\left(\frac{a}{c}\right) + \operatorname{st}\left(\frac{b}{c}\right)$ provided the right hand side addition is defined;
5. if $c > 0$ and $0 \leq a \leq b$ then $\operatorname{st}\left(\frac{a}{c}\right) \leq \operatorname{st}\left(\frac{b}{c}\right)$;
6. if $\operatorname{st}\left(\frac{a}{b}\right) \notin \{0, \pm\infty\}$ then $\operatorname{st}\left(\frac{a}{b}\right) = \operatorname{st}\left(\frac{b}{a}\right)^{-1}$.

Proof. We will consider each of the statements individually.

1. We prove the case for $0 < a, b, c \in \Gamma$. The other cases following in a similar manner and the result extends trivially to $a, b, c \in \Gamma/\mathbb{Z}$.

   So we wish to show that $\mathrm{st}\left(\frac{a}{b}\right) \cdot \mathrm{st}\left(\frac{b}{c}\right) \subseteq \mathrm{st}\left(\frac{a}{c}\right)$. Let $q \in \mathrm{st}\left(\frac{a}{b}\right) \cdot \mathrm{st}\left(\frac{b}{c}\right)$, so that $q = q_1 \cdot q_2$ for some $q_1, q_2 \in \mathbb{Q}$ such that there exist $r_1, r_2, s_1, s_2 \in \mathbb{N}$ with $s_1, s_2 > 0, \frac{r_1}{s_1} > q_1, \frac{r_2}{s_2} > q_2, r_1 b < s_1 a$ and $r_2 c < s_2 b$. Set $r = r_1 r_2$ and $s = s_1 s_2$. Then

$$\frac{r}{s} = \frac{r_1 r_2}{s_1 s_2} = \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} > q_1 \cdot q_2 = q$$

   and

$$rc = r_1 r_2 c < r_1 s_2 b < s_1 s_2 a < sa.$$

   Thus $q \in \mathrm{st}\left(\frac{a}{c}\right)$ as required.

   We must now show that $\mathrm{st}\left(\frac{a}{c}\right) \subseteq \mathrm{st}\left(\frac{a}{b}\right) \cdot \mathrm{st}\left(\frac{b}{c}\right)$. Letting $q \in \mathrm{st}\left(\frac{a}{c}\right)$ we wish to find $q_1 \in \mathrm{st}\left(\frac{a}{b}\right)$ and $q_2 \in \mathrm{st}\left(\frac{b}{c}\right)$ so that $q = q_1 \cdot q_2$. If we set $k = \mathrm{st}\left(\frac{a}{c}\right)$ and $k_1 = \mathrm{st}\left(\frac{a}{b}\right)$ then by the usual definition of an extended cut we know that for all $x \in \mathbb{Q}$ we have $x \in \mathrm{st}\left(\frac{a}{c}\right)$ iff $x < k$ and similarly $x \in \mathrm{st}\left(\frac{a}{b}\right)$ iff $x < k_1$.

   Now since $q \in \mathrm{st}\left(\frac{a}{c}\right)$ we know that there exists some $\frac{r}{s}$ with $q < \frac{r}{s} < k$ so that $rc < sa$. For some $\delta \in \mathbb{Q}$ with $0 < \delta \leq 1$ we therefore have that $q < (1-\delta)\frac{r}{s}$. Set $\frac{u}{v}$ to be some element in the range $k_1 < \frac{u}{v} < (1+\delta/4)k_1$ and q_1 to be some element in the range $(1-\delta/2)k_1 < q_1 < k_1$. Clearly $q_1 \in \mathrm{st}\left(\frac{a}{b}\right)$. Set $\frac{r_2}{s_2} = \frac{rv}{su}$. Since $\frac{u}{v} \notin \mathrm{st}\left(\frac{a}{b}\right)$ we know that $va \leq ub$. It is therefore the case that $rvc < sva \leq sub$ and hence $r_2 c < s_2 b$. Letting $q_2 = \frac{q}{q_1}$, if $q_2 < \frac{r_2}{s_2}$ we will therefore have that $q_2 \in \mathrm{st}\left(\frac{b}{c}\right)$ which will give us our result. Now $q < (1-\delta)\frac{r}{s}$ and $q_1 > (1-\delta/2)k_1$. Moreover $\frac{u}{v} < (1+\delta/4)k_1$ and so

$$q_1 \cdot \frac{v}{u} > \frac{(1-\delta/2)}{(1+\delta/4)} \cdot \frac{k_1}{k_1},$$

   which gives us the result that

$$\begin{aligned} q_1 &> \frac{(1+\delta/4-3\delta/4)}{(1+\delta/4)} \cdot \frac{u}{v} \\ &= \left(1 - \frac{3\delta}{4+\delta}\right)\frac{u}{v} \\ &> \left(1 - \frac{3}{4}\delta\right)\frac{u}{v} \\ &> (1-\delta)\frac{u}{v}. \end{aligned}$$

So

$$q_2 = \frac{q}{q_1} < \frac{(1-\delta)\frac{r}{s}}{(1-\delta)\frac{u}{v}} = \frac{rv}{su} = \frac{r_2}{s_2}$$

and we do indeed have $q_2 \in \text{st}\left(\frac{b}{c}\right)$ as required.

2. We give the case for $q > 0$, with the case $q < 0$ following similarly and the case $q = 0$ being trivial. So let $q = \frac{n_1}{n_2} > 0$ for some $n_1, n_2 \in \mathbb{N}$. We will show the result for $a, b \in \Gamma$, so that taking

$$\text{st}\left(\frac{q(\mathbb{Z}+a)}{\mathbb{Z}+b}\right) = \text{st}\left(\frac{n_1 a}{n_2 b}\right)$$

we can extend the result trivialy to elements of $\Gamma/\mathbb{Z}$. We first show that

$$\text{st}\left(\frac{n_1 a}{n_2 b}\right) \subseteq q \cdot \text{st}\left(\frac{a}{b}\right).$$

So suppose $q_1 \in \text{st}\left(\frac{n_1 a}{n_2 b}\right)$. Then $\exists\, r, s \in \mathbb{N}, s > 0$ with $q_1 < \frac{r}{s}$ and $rn_2 b < sn_1 a$. We claim that $q_1 \cdot \frac{n_2}{n_1} \in \text{st}\left(\frac{a}{b}\right)$.

Certainly $q_1 < \frac{r}{s} \quad \Rightarrow \quad q_1 \cdot \frac{n_2}{n_1} < \frac{rn_2}{sn_1}$, and $rn_2 b < sn_1 a$. So indeed $\frac{q_1}{q} \in \text{st}\left(\frac{a}{b}\right)$, so that $q_1 \in q \cdot \text{st}\left(\frac{a}{b}\right)$ as required.

We must now show that

$$q \cdot \text{st}\left(\frac{a}{b}\right) \subseteq \text{st}\left(\frac{n_1 a}{n_2 b}\right).$$

So suppose $q_1 \in q \cdot \text{st}\left(\frac{a}{b}\right)$. Then $\frac{q_1}{q} \in \text{st}\left(\frac{a}{b}\right)$ so $\exists\, r, s \in \mathbb{N}, s > 0$ with $\frac{q_1}{q} < \frac{r}{s}$ and $rb < sa$. So $q_1 < \frac{rn_1}{sn_2}$ and $rn_1 n_2 b < sn_1 n_2 a$. Let $r' = rn_1$ and $s' = sn_2$. Then

$$q_1 < \frac{r'}{s'} \text{ and } r' n_2 b < s' n_1 a.$$

Thus $q_1 \in \text{st}\left(\frac{n_1 a}{n_2 b}\right) = \text{st}\left(\frac{q(\mathbb{Z}+a)}{\mathbb{Z}+b}\right)$ as required.

3. By letting $q = \frac{n_2}{n_1}$ we see that this result follows directly from the proof of (2) above.

4. Again we take $a, b, c \in \Gamma$, with the result extending trivially to elements of $\Gamma/\mathbb{Z}$. So we start by showing that

$$\text{st}\left(\frac{a}{c}\right) + \text{st}\left(\frac{b}{c}\right) \subseteq \text{st}\left(\frac{a+b}{c}\right).$$

Suppose $q_1 \in \text{st}\left(\frac{a}{c}\right), q_2 \in \text{st}\left(\frac{b}{c}\right)$ and $q = q_1 + q_2$.

$$\begin{aligned} \text{Then } \exists r_1, s_1 \in \mathbb{N}, s_1 > 0 \text{ with } q_1 < \frac{r_1}{s_1} \text{ and } r_1 c < s_1 a; \\ \exists r_2, s_2 \in \mathbb{N}, s_2 > 0 \text{ with } q_2 < \frac{r_2}{s_2} \text{ and } r_2 c < s_2 b. \end{aligned}$$

So

$$q_1 + q_2 < \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}.$$

Also, $r_1 s_2 c < s_1 s_2 a$ and $r_2 s_1 c < s_1 s_2 b$, making $(r_1 s_2 + r_2 s_1)c < s_1 s_2 (a+b)$. So $q = q_1 + q_2 \in \text{st}\left(\frac{a+b}{c}\right)$ as required.

We must now show that

$$\text{st}\left(\frac{a+b}{c}\right) \subseteq \text{st}\left(\frac{a}{c}\right) + \text{st}\left(\frac{b}{c}\right).$$

Let $k = \text{st}\left(\frac{a+b}{c}\right), k_1 = \text{st}\left(\frac{a}{c}\right)$ and $k_2 = \text{st}\left(\frac{b}{c}\right)$ where k, k_1, k_2 are the reals identified with the sets of rationals in each case. Further let $q \in \text{st}\left(\frac{a+b}{c}\right)$ so that $q < k$. From this we know that $\exists r, s \in \mathbb{N}, s > 0$ with $q < \frac{r}{s}$ and $rc < s(a+b)$. We may assume without loss of generality that $\frac{r}{s} \neq k$ and must find $q_1 \in \text{st}\left(\frac{a}{c}\right)$ and $q_2 \in \text{st}\left(\frac{b}{c}\right)$ so that $q = q_1 + q_2$.

Set $\epsilon = \min\left\{k - \frac{r}{s}, \frac{r}{s} - q\right\}$ and choose $q_1 \in \mathbb{Q}, r_1, s_1 \in \mathbb{N}$ so that

$$k_1 - \epsilon < q_1 < \frac{r_1}{s_1} < k_1, \tag{8.1}$$

where $r_1 c < s_1 a$. Clearly $q_1 \in \text{st}\left(\frac{a}{c}\right)$. We set $q_2 = q - q_1$ and claim that $q_2 \in \text{st}\left(\frac{b}{c}\right)$. Once we have shown this the proof will be complete.

From the inequality (8.1) we have that $k_1 - q_1 < \epsilon$. So

$$\begin{aligned} & \frac{r_1}{s_1} - q_1 < \epsilon \\ \Rightarrow\quad & \frac{r_1}{s_1} - q_1 < \frac{r}{s} - q \\ \Rightarrow\quad & q_2 = q - q_1 < \frac{r}{s} - \frac{r_1}{s_1} = \frac{r s_1 - r_1 s}{s s_1} \end{aligned}$$

Setting $r_2 = rs_1 - r_1 s$ and $s_2 = ss_1$ we therefore have $q_2 < \frac{r_2}{s_2}$. It remains to show that $r_2 c < s_2 b$.

By (8.1), $k_1 - \frac{r_1}{s_1} < \epsilon$ so $k_1 - \frac{r_1}{s_1} < k - \frac{r}{s}$. Alternatively we can write

$$\text{st}\left(\frac{a}{c}\right) - \frac{r_1}{s_1} < \text{st}\left(\frac{a+b}{c}\right) - \frac{r}{s},$$

and so we are able to find some $q' \in \mathbb{Q}$ so that

$$\mathrm{st}\left(\frac{a}{c}\right) - \frac{r_1}{s_1} < q' < \mathrm{st}\left(\frac{a+b}{c}\right) - \frac{r}{s}.$$

By the definition of the standard parts we see that this means that

$$q' + \frac{r_1}{s_1} \notin \mathrm{st}\left(\frac{a}{c}\right) \text{ whilst } q' + \frac{r}{s} \in \mathrm{st}\left(\frac{a+b}{c}\right). \tag{8.2}$$

The latter of these two results tells us that for some $n_1, n_2 \in \mathbb{N}, n_2 > 0$ we have $q' < \frac{n_1}{n_2}$ giving

$$q' + \frac{r}{s} < \frac{n_1}{n_2} + \frac{r}{s} = \frac{n_1 s + n_2 r}{n_2 s}$$

and

$$(n_1 s + n_2 r)c < n_2 s(a+b).$$

In particular we have

$$s_1(n_1 s + n_2 r)c < n_2 s s_1(a+b). \tag{8.3}$$

The first result of (8.2) tells us that for every $x, y \in \mathbb{N}, y > 0$ for which $\frac{x}{y} > q' + \frac{r_1}{s_1}$ we must have $xc > ya$. Hence as

$$\frac{n_1}{n_2} + \frac{r_1}{s_1} > q' + \frac{r_1}{s_1}$$

we must therefore have that $(n_1 s_1 + n_2 r_1)c > n_2 s_1 a$. In particular we have

$$-s(n_1 s_1 + n_2 r_1)c < -s n_2 s_1 a. \tag{8.4}$$

From the inequalities (8.3) and (8.4) we deduce that

$$s_1(n_1 s + n_2 r)c - s(n_1 s_1 + n_2 r_1)c < n_2 s s_1 b + n_2 s s_1 a - n_2 s s_1 a,$$

which on cancelling gives

$$(r s_1 - r_1 s)c < s s_1 b$$

and hence our result

$$r_2 c < s_2 b.$$

5. Suppose that $a, b \in \Gamma$ with $a \leq b$. Then we must show that $\operatorname{st}\left(\frac{a}{c}\right) \subseteq \operatorname{st}\left(\frac{b}{c}\right)$. If $q \in \operatorname{st}\left(\frac{a}{c}\right)$ then for some $\frac{r}{s}$ we know that $q < \frac{r}{s}$ and $rc < sa$. But $a \leq b$ so $sa \leq sb$ and hence $rc < sb$. It therefore follows directly that $q \in \operatorname{st}\left(\frac{b}{c}\right)$, giving us the required result that $\operatorname{st}\left(\frac{a}{c}\right) \subseteq \operatorname{st}\left(\frac{b}{c}\right)$. Again, this result extends trivially to elements of $\Gamma/\mathbb{Z}$.

6. This result follows immediately from (1) where we take $c = a$.

□

8.2 Strong independence

Definition 8.2.1. We say that $B \subseteq \Gamma/\mathbb{Z}$ is **strongly independent** if $0 \notin B$ and

$$\left\{ \operatorname{st}\left(\frac{b}{a}\right) : b \in B \right\} \setminus \{0, \pm\infty\}$$

is linearly independent over $\mathbb{Q}$ as a subset of $\mathbb{R}$, for all $a \in \Gamma/\mathbb{Z}$.

We say $B \subseteq \Gamma$ is **strongly independent** if it contains at most one representative of each coset $\mathbb{Z} + a$, with $B/\mathbb{Z}$ strongly independent as defined above.

In order to make good use of the definition of strong independence we must consider the structure of $\widetilde{\Gamma} = \Gamma/\mathbb{Z}$ further. Our aim in the next chapter is to show that we can, with certain conditions, extend maps between pairs of strongly independent sets to automorphisms defined on the entire Presburger group to which these sets belong. In order to achieve this we introduce some new definitions:

If $a, b \in \widetilde{\Gamma}$ we set $a \equiv b$ to mean that either

$$\begin{aligned} & a = b = 0 \\ \text{or} \quad & a, b \neq 0 \quad \text{and} \quad \operatorname{st}\left(\frac{a}{b}\right) \notin \{0, \pm\infty\}. \end{aligned}$$

By lemma 8.1.5, this is an equivalence relation, and so we can define the following:

Definition 8.2.2. We call $\mathrm{V} = \widetilde{\Gamma}/\equiv$ the set of **values** of $\widetilde{\Gamma}$.

V is linearly ordered by

$$a/\equiv \, < b/\equiv \quad \Longleftrightarrow \quad a/\equiv \, \neq b/\equiv \text{ and } |a| < |b|.$$

The **valuation map** $\mathrm{v} \colon \widetilde{\Gamma} \to \mathrm{V}$ is defined by

$$a \mapsto a/\equiv.$$

Proposition 8.2.3. The valuation map v has the following properties:

1. $\mathrm{v}(qa) = \mathrm{v}(a)$ for all $q \in \mathbb{Q} \setminus \{0\}$;
2. if $|a| \leq |b|$ then $\mathrm{v}(a) \leq \mathrm{v}(b)$;
3. if $n|a| < |b|$ for all $n \in \mathbb{N}$ then $\mathrm{v}(a) < \mathrm{v}(b)$;
4. $\mathrm{v}(a+b) \leq \max(\mathrm{v}(a), \mathrm{v}(b))$ with equality unless $\mathrm{v}(a) = \mathrm{v}(b)$;
5. if $\mathrm{v}(a) = \mathrm{v}(b)$ then $\mathrm{v}(a+b) < \mathrm{v}(a), \mathrm{v}(b)$ for $\mathrm{st}\left(\frac{a}{b}\right) = -1$ and if $\mathrm{st}\left(\frac{a}{b}\right) \neq -1$ then $\mathrm{v}(a) = \mathrm{v}(b) = \mathrm{v}(a+b)$.

Property 4 tells us that the valuation is non-Archimedean. The form of this valuation is not entirely conventional, being in some sense a 'reversal' of the usual format, which would require that

$$\mathrm{v}(a+b) \geq \min(\mathrm{v}(a), \mathrm{v}(b))$$

(see for example [44, p. 73]) and also that $\mathrm{v}(0) = \infty$, whilst from our definition 8.2.2 above we find that $\mathrm{v}(0) = 0$. We have chosen this reversed form for simplicity in the context and it is highlighted here only to avoid confusion; for a real valuation the two forms are interchangeable by taking inverses.

As we have seen, multiplication in $\widetilde{\Gamma}$ in the general case is not defined – only multiplication by rationals – and so in common with real valuations, properties 1, 2 and 4 also tell us that for $a \in \mathbb{Q}$

$$\mathrm{v}(ab) = \mathrm{v}(b) = \mathrm{v}(0) + \mathrm{v}(b) = \mathrm{v}(a) + \mathrm{v}(b).$$

We choose for each $\upsilon \in \mathrm{V}$ a positive representative $\gamma_\upsilon \in \widetilde{\Gamma}$ such that $\mathrm{v}(\gamma_\upsilon) = \upsilon$. In considering the valuation map, we will identify υ with γ_υ, thinking of the image of $\mathrm{v}\colon \widetilde{\Gamma} \to \{\, \gamma_\upsilon : \upsilon \in \mathrm{V}\}$ as being 'equal' to V. We will also use v to refer to the mapping $\mathrm{v}\colon \Gamma \to \Gamma/\equiv$ defined with equivalent conditions. Although it will usually be clear from the context which of the two we are considering, the maps are in effect identical and we will not distinguish between them.

Classically these valuation classes also occur in another very natural way when we consider convex submodels and the embeddings which they produce. We define all of these convex submodels in the following way:

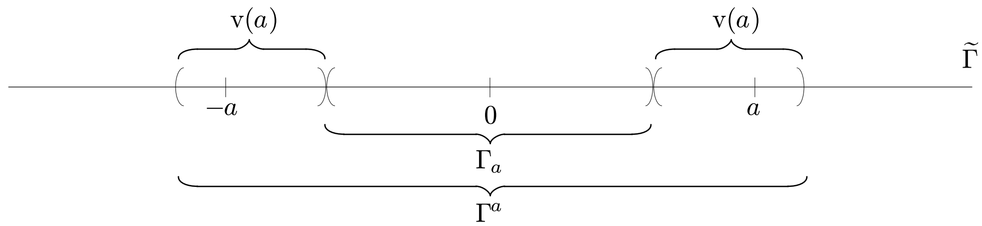

Figure 8.2: The relationship between Γ_a, Γ^a and $\mathrm{v}(a)$.

Definition 8.2.4. Given $a \in \widetilde{\Gamma}$, we define Γ_a and Γ^a to be as follows:

$$\Gamma_a = \left\{ b \in \widetilde{\Gamma} : \mathrm{st}\left(\frac{b}{a}\right) = 0 \right\}, \quad \Gamma^a = \left\{ b \in \widetilde{\Gamma} : \mathrm{st}\left(\frac{b}{a}\right) \in \mathbb{R} \right\}.$$

By lemma 8.1.5 these are both convex divisible subgroups of $\widetilde{\Gamma}$ with $\Gamma_a \leq \Gamma^a \leq \widetilde{\Gamma}$. Moreover, Γ^a/Γ_a is also divisible and the map $b \mapsto \mathrm{st}\left(\frac{b}{a}\right)$ is a group homomorphism $\Gamma^a \to \mathbb{R}$ preserving $<$, with kernel Γ_a. It follows that Γ^a/Γ_a is isomorphic to a divisible ordered subgroup of $(\mathbb{R}; +, <)$.

The sets Γ_a and Γ^a are actually designated by the valuation class of a with $\mathrm{v}(a) = \Gamma^a \setminus \Gamma_a$. This is because

$$\Gamma_a = \{\, x : \forall n \in \mathbb{N}\ n|x| < |a| \} = \{\, x : \mathrm{v}(x) < \mathrm{v}(a) \} = \{\, x : \mathrm{st}\left(\frac{x}{a}\right) = 0 \},$$

and

$$\Gamma^a = \{\, x : \exists n \in \mathbb{N}\ n|a| > |x| \} = \{\, x : \mathrm{v}(x) \leq \mathrm{v}(a) \} = \{\, x : \mathrm{st}\left(\frac{x}{a}\right) \in \mathbb{R} \}.$$

It's also clear from this that for $a, b \in \widetilde{\Gamma}$ we have $\Gamma_a = \Gamma_b$ and $\Gamma^a = \Gamma^b$ if and only if $a \equiv b$. The situation is depicted in figure 8.2.

We will extend our notation so that if $v \in \mathrm{V}$ then $\Gamma_v \stackrel{\mathrm{def}}{=} \Gamma_a$ where $\mathrm{v}(a) = v$ and similarly for Γ^v. By the immediately preceding remarks this notation is well defined.

By ordering $\bigoplus_{a \in \mathrm{V}} \Gamma^a/\Gamma_a$ lexicographically it is possible to produce an embedding of ordered groups

$$\bigoplus_{a \in \mathrm{V}} \Gamma^a/\Gamma_a \to \widetilde{\Gamma}.$$

We can also find an embedding into a restriction of the cartesian product known as the **Hahn product**:

$$\widetilde{\Gamma} \to \prod_{a \in \mathrm{V}}{}^{\mathrm{H}} \Gamma^a/\Gamma_a,$$

giving us the sequence of embeddings

$$\bigoplus_{a \in \mathrm{V}} \Gamma^a/\Gamma_a \to \widetilde{\Gamma} \to \prod_{a \in \mathrm{V}}{}^{\mathrm{H}} \Gamma^a/\Gamma_a \to \prod_{a \in \mathrm{V}} \Gamma^a/\Gamma_a.$$

The Hahn product is defined as those elements of the cartesian product for which the support is well-ordered by the reverse of the standard ordering on the valuations V derived from Γ.

All of these results are well known and we have provided full proofs in the appendix (§14.1). It is hoped that further refinements or restrictions of the Hahn product will give further information and that this could provide the basis for further work in this area.

We now return to our original investigation of valuations. The most important results concerning the relationship between standard parts and valuations as outlined above are given by the following two lemmas.

Lemma 8.2.5. The set $B \subseteq \widetilde{\Gamma}$ is strongly independent if and only if $0 \notin B$ and every nontrivial $\mathbb{Q}$-linear combination

$$a = q_1 b_1 + \cdots + q_n b_n$$

has value

$$\mathrm{v}(a) = \max\{\, \mathrm{v}(b_j) : 1 \le j \le n, q_j \ne 0\}$$

where $\bar{q} \in \mathbb{Q}$ and $\bar{b} \in B$.

Proof. Suppose B is strongly independent with $b_1, \ldots, b_n \in B$ and $q_1, \ldots, q_n \in \mathbb{Q}\backslash\{0\}$. Put $a = \max\{\, |b_j| : 1 \le j \le n, q_j \ne 0\}$. By definition of strong independence we know that $b_1, \ldots, b_n \ne 0$ and so in particular we know that $a \ne 0$. Then

$$\mathrm{st}\left(\frac{\sum_j q_j b_j}{a}\right) = \sum_j q_j \,\mathrm{st}\left(\frac{b_j}{a}\right)$$

and by strong independence this is non-zero. Hence by definition of $\mathrm{v}(a)$

$$\mathrm{v}\left(\sum_j q_j b_j\right) = \mathrm{v}(a)$$

as required.

Conversely, suppose $B \subseteq \widetilde{\Gamma}$ satisfies the property that every nontrivial $\mathbb{Q}$-linear combination $a = q_1 b_1 + \cdots + q_n b_n$ with $\bar{q} \in \mathbb{Q}$ and $\bar{b} \in B$ has value $\mathrm{v}(a) = \max\{\, \mathrm{v}(b_j) : 1 \le j \le n, q_j \ne 0\}$. Take $b_1, \ldots, b_n \in B$ and $a \in \widetilde{\Gamma}$. Then

$$\left\{\, b_i : \mathrm{st}\left(\frac{b_i}{a}\right) \notin \{0, \pm\infty\} \right\} = \{\, b_i : \mathrm{v}(b_i) = \mathrm{v}(a)\}\,,$$

so it suffices to show that the set

$$\left\{\text{st}\left(\frac{b_1}{a}\right), \ldots, \text{st}\left(\frac{b_n}{a}\right)\right\}$$

is independent for $b_i \in B$ with $\text{v}(b_i) = \text{v}(a)$. But for such b_i, if

$$\sum_i q_i \,\text{st}\left(\frac{b_i}{a}\right) = \text{st}\left(\frac{\sum_i q_i b_i}{a}\right) = 0$$

then $\text{v}\left(\sum_i q_i b_i\right) < \text{v}(a)$, which contradicts the property unless all the q_i are zero. □

Corollary 8.2.6. Every strongly independent set B is linearly independent.

Proof. Suppose that B is strongly independent but not linearly independent. We aim to find a contradiction. So for some $b_1, \ldots, b_n \in B$ and $q_1, \ldots, q_n \in \mathbb{Q}$ we suppose we have that

$$a = q_1 b_1 + \cdots + q_n b_n = 0.$$

Without loss of generality we may assume that the q_i's are all non-zero. But $\text{v}(a) = 0$, so by strong independence and lemma 8.2.5 it follows that $\max\{\text{v}(b_1), \ldots, \text{v}(b_n)\} = 0$. So $\text{v}(b_1) = \text{v}(b_1) = \cdots = \text{v}(b_n) = 0$ which implies that $b_1 = b_2 = \cdots = b_n = 0$ (since $0 = \mathbb{Z} \in \widetilde{\Gamma}$) contradicting the definition of strong independence. □

Proposition 8.2.7. For $\gamma_1, \gamma_2 \in \widetilde{\Gamma}$ the set $\{\gamma_1, \gamma_2\}$ is strongly independent if and only if $\text{st}\left(\frac{\gamma_1}{\gamma_2}\right) \notin \mathbb{Q}$. If $\text{st}\left(\frac{\gamma_1}{\gamma_2}\right) = q \in \mathbb{Q}$ then $\text{v}(\gamma_1 - q\gamma_2) < \text{v}(\gamma_1) = \text{v}(\gamma_2)$.

Proof. First suppose that $\{\gamma_1, \gamma_2\}$ is not strongly independent. Then for some $q_1, q_2 \in \mathbb{Q}$ we know by lemma 8.2.5 that $\text{v}(q_1\gamma_1 + q_2\gamma_2) < \max\{\, \text{v}(\gamma_i) : i = 1, 2, q_i \neq 0\}$. In fact, it is trivially the case that we cannot have either $q_1 = 0$ or $q_2 = 0$ and we also know that $\text{v}(\gamma_1) = \text{v}(\gamma_2)$, hence we must have $\text{v}(q_1\gamma_1 + q_2\gamma_2) < \text{v}(\gamma_2)$. But then

$$\text{st}\left(\frac{q_1\gamma_1 + q_2\gamma_2}{\gamma_2}\right) = 0$$

from which

$$q_1.\,\text{st}\left(\frac{\gamma_1}{\gamma_2}\right) + q_2.\,\text{st}\left(\frac{\gamma_2}{\gamma_2}\right) = q_1.\,\text{st}\left(\frac{\gamma_1}{\gamma_2}\right) + q_2 = 0$$

and so

$$\text{st}\left(\frac{\gamma_1}{\gamma_2}\right) = \frac{-q_2}{q_1} \in \mathbb{Q}$$

as required.

For the reverse direction, suppose that $\mathrm{st}\left(\frac{\gamma_1}{\gamma_2}\right) = q \in \mathbb{Q}$. Then the set

$$\left\{\mathrm{st}\left(\frac{\gamma_1}{\gamma_2}\right), \mathrm{st}\left(\frac{\gamma_2}{\gamma_2}\right)\right\} = \{q, 1\}$$

is clearly not linearly independent over $\mathbb{Q}$ as a subset of $\mathbb{R}$, and it follows from the definition of strong independence (8.2.1) that the set $\{\gamma_1, \gamma_2\}$ is not strongly independent.

For the final part, we know that $\mathrm{v}(q_1\gamma_1 + q_2\gamma_2) < \mathrm{v}(\gamma_2)$ and hence that $\mathrm{v}(\gamma_1 + \frac{q_2}{q_1}\gamma_2) < \mathrm{v}(q_1^{-1}\gamma_2) = \mathrm{v}(\gamma_2)$. We therefore know that

$$\mathrm{st}\left(\frac{\gamma_1}{\gamma_2}\right) = \frac{-q_2}{q_1} = q$$

and so $\mathrm{v}(\gamma_1 - q\gamma_2) = \mathrm{v}(\gamma_1 + \frac{q_2}{q_1}\gamma_2) < \mathrm{v}(\gamma_1)$ as required. □

Lemma 8.2.8. Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in \widetilde{\Gamma}$ are such that $\mathrm{v}(a_i) \neq \mathrm{v}(b_j)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$. Then the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are each strongly independent if and only if $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ is.

Proof. Suppose the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are each strongly independent but $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ is not. Then for some $q_1, \ldots, q_n, q'_1, \ldots, q'_n \in \mathbb{Q}$ we have

$$q_1a_1 + \cdots + q_na_n + q'_1b_1 + \cdots + q'_nb_n = c$$

where $c \in \widetilde{\Gamma}$ is such that

$$\mathrm{v}(c) < \max\left(\{\, \mathrm{v}(a_i) : 1 \leq i \leq n, q_i \neq 0\} \cup \{\, \mathrm{v}(b_i) : 1 \leq i \leq n, q'_i \neq 0\}\right).$$

Since $\mathrm{v}(a_i) \neq \mathrm{v}(b_j)$ for any $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$ we know that this maximum will be achieved either as some a_i or some b_i, but not both. Hence by rearranging and without loss of generality we may suppose that this maximum takes the value of a_1 and that $a_1, \ldots, a_j$ are the only elements which achieve the maximum value for $j \in \mathbb{N}$ with $1 \leq j \leq n$. Then

$$(q_1a_1 + \cdots + q_ja_j) + q_{j+1}a_{j+1} + \cdots + q_na_n + q'_1b_1 + \cdots + q'_nb_n = c,$$

and so

$$q_1a_1 + \cdots + q_ja_j = c - (q_{j+1}a_{j+1} + \cdots + q_na_n + q'_1b_1 + \cdots + q'_nb_n).$$

But then $\mathrm{v}(q_1a_1 + \cdots + q_ja_j) < \mathrm{v}(a_1) = \cdots = \mathrm{v}(a_j)$, contradicting the fact that the set $\{a_1, \ldots, a_n\}$ is strongly independent. This gives us one direction; the opposite direction is trivially the case. □

Definition 8.2.9. If $a_1, \dots, a_n \in \widetilde{\Gamma}$ then the **spanning set**, written as $\langle a_1, \dots, a_n \rangle$, is the smallest subset of $\widetilde{\Gamma}$ closed under the operations of addition and scalar multiplication. Hence

$$\langle a_1, \dots, a_n \rangle = \Big\{ \sum_{i=1}^{n} q_i a_i : q_1, \dots, q_n \in \mathbb{Q} \Big\}.$$

Lemma 8.2.10 (Exchange Lemma). If $a_1, \dots, a_n$ are strongly independent in $\widetilde{\Gamma}$, and $a \in \widetilde{\Gamma}$ then

$$\begin{aligned} &\text{either } a \in \langle a_1, \dots, a_n \rangle \\ &\quad \text{or } \exists\, a_{n+1} \in \langle a_1, \dots, a_n, a \rangle \text{ such that } a_1, \dots, a_n, a_{n+1} \\ &\qquad \text{are strongly independent and } a \in \langle a_1, \dots, a_n, a_{n+1} \rangle . \end{aligned}$$

Proof. We prove this by induction on the set

$$\chi = \operatorname{card} |\{ \mathrm{v}(a_i) : \mathrm{v}(a_i) \leq \mathrm{v}(a) \}| .$$

If $\chi = 0$ and $a \neq 0$ then the set $\{a_1, \dots, a_n, a\}$ is strongly independent by definition (cf. 8.2.1). So suppose that $\chi > 0$ and $\{a_1, \dots, a_n, a\}$ is linearly independent. If it is strongly independent then we can simply take $a_{n+1} = a$ and we are done. Otherwise, by lemma 8.2.5 there exist $q_i \in \mathbb{Q}$ for all $i \in \mathbb{N}$ such that $1 \leq i \leq n$, and $q \in \mathbb{Q} \setminus \{0\}$ for which

$$a' = \sum_{i=1}^{n} q_i a_i + qa$$

has value

$$\mathrm{v}(a') < \max \{ \{ \mathrm{v}(a_i) : 1 \leq i \leq n, q_i \neq 0 \} \cup \{ \mathrm{v}(a) \} \} . \tag{8.5}$$

Now we intend to show that the following hold:-

1. all a_i with $q_i \neq 0$ have value $\mathrm{v}(a_i) \leq \mathrm{v}(a)$ and
2. there is at least one such a_i for which $\mathrm{v}(a_i) = \mathrm{v}(a)$.

We tackle them individually.

1. Suppose that this were not the case. Then by rearranging the elements if necessary we may suppose that for some $r \in \mathbb{N}$ with $1 \leq r \leq n$ we have that $\mathrm{v}(a_1), \dots, \mathrm{v}(a_r) > \mathrm{v}(a)$ whilst $\mathrm{v}(a_{r+1}), \dots, \mathrm{v}(a_n) \leq \mathrm{v}(a)$, with at least one of $q_1, \dots, q_r$ not equal to 0. But then

$$\mathrm{v}(q_1 a_1 + \dots + q_r a_r) \geq \min\{ \mathrm{v}(a_i) : i \leq r \} > \mathrm{v}(a),$$

and so

$$\mathrm{v}\left(\sum_{i=1}^{n} q_i a_i + qa\right) = \mathrm{v}\left(\sum_{i=1}^{r} q_i a_i\right) > \mathrm{v}(a)$$

since every other a_i has value less than or equal to that of a. From equation (8.5) and our assumption that $\mathrm{v}(a)$ is not maximal we see that $\mathrm{v}\left(\sum_{i=1}^{r} q_i a_i\right) = \mathrm{v}(a') < \max\{\, \mathrm{v}(a_i) : 1 \leq i \leq r, q_i \neq 0\}$. But referring to lemma 8.2.5 we see that this contradicts the fact that $a_1, \ldots, a_n$ are strongly independent.

2. Again, suppose that this is not the case. Then $\mathrm{v}(a_i) < \mathrm{v}(a)$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$ and $q_i \neq 0$, and so

$$\mathrm{v}\left(\sum_{i=1}^{n} q_i a_i + qa\right) = \mathrm{v}(qa) = \mathrm{v}(a) = \max\left\{\{\, \mathrm{v}(a_i) : 1 \leq i \leq n, q_i \neq 0\} \cup \{\mathrm{v}(a)\}\right\}$$

which contradicts our assumption of equation (8.5).

By 1, 2 above, $\chi' = \mathrm{card}\,|\{\, \mathrm{v}(a_i) : \mathrm{v}(a_i) \leq \mathrm{v}(a')\}| < \chi$. Also, $a_1, \ldots, a_n, a'$ are linearly independent, since $q \neq 0$. So by our inductive hypothesis there is some a_{n+1} with $a_1, \ldots, a_n, a_{n+1}$ strongly independent and $a' \in \langle a_1, \ldots a_n, a_{n+1}\rangle$ as required. □

A crucial connection between types and standard parts is provided by the following lemma. Later on we will produce a stronger specialisation of the implication from (1) to (2), which can be found as theorem 9.3.8 on page 79.

Lemma 8.2.11. Suppose Γ is a model of Presburger arithmetic, and $\bar{x}, \bar{y} \in \Gamma^n$ are such that $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are both strongly independent sets. Then the following are equivalent :

1. $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$;
2. $\varrho(x_i) = \varrho(y_i)$ and $\mathrm{st}\left(\frac{x_i}{x_j}\right) = \mathrm{st}\left(\frac{y_i}{y_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i \leq j \leq n$.

Proof. (1) $\Longrightarrow$ (2) Suppose $x_i \equiv r \pmod m$ for some $m \in \mathbb{N}$ and $0 \leq r < m$. Now '$x_i \equiv r \pmod m$' is a basic formula and so obviously

$$\text{'}x_i \equiv r \pmod m\text{'} \in \mathrm{tp}(\bar{x}) \iff \text{'}y_i \equiv r \pmod m\text{'} \in \mathrm{tp}(\bar{y})\,.$$

It clearly follows from this that $\varrho(x_i) = \varrho(y_i)$.

Also, $\mathrm{st}\left(\frac{x_i}{x_j}\right) = \left\{\, q \in \mathbb{Q} : q < \frac{r}{s} \text{ for some } r, s \in \mathbb{Z} \text{ and } rx_j < sx_i\right\}$. So suppose $q \in \mathrm{st}\left(\frac{x_i}{x_j}\right)$. Then for some $r, s \in \mathbb{Z}$ we have $q < \frac{r}{s}$ and $rx_j < sx_i$. But then '$sx_i - rx_j > 0$'

is a basic formula, so occurs in $\mathrm{tp}(\bar{x})$. Therefore '$sy_i - ry_j > 0$' occurs in $\mathrm{tp}(\bar{y})$. Clearly then $q \in \mathrm{st}\left(\frac{y_i}{y_j}\right)$ as well. By symmetry, we must have $\mathrm{st}\left(\frac{x_i}{x_j}\right) = \mathrm{st}\left(\frac{y_i}{y_j}\right)$.

It should be noted that in this part of the proof it has not been necessary to use the strong independence of the sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$.

(2) $\implies$ (1) By Quantifier Elimination (theorem 4.2.1) we only need to show that the basic formulas of $\mathrm{tp}(\bar{x})$ and $\mathrm{tp}(\bar{y})$ are the same. Now $\varrho(x_i) = \varrho(y_i)$, so clearly for any $i \in \mathbb{N}$ with $1 \le i \le n$ and $m \in \mathbb{N}$ we have

$$\text{`}x_i \equiv r \pmod{m}\text{'} \in \mathrm{tp}(\bar{x}) \iff \text{`}y_i \equiv r \pmod{m}\text{'} \in \mathrm{tp}(\bar{y})\,.$$

Now suppose

$$\text{`}\sum_{i=1}^{n} \gamma_i x_i > 0\text{'} \in \mathrm{tp}(\bar{x})$$

where $\gamma_i \in \mathbb{Z}$ for all $i \in \mathbb{N}$ with $1 \le i \le n$. Let j be such that $\mathrm{v}(x_j) = \max\{\,\mathrm{v}(x_i) : 1 \le i \le n \text{ with } \gamma_i \ne 0\}$. Then $\mathrm{st}\left(\frac{x_i}{x_j}\right) \notin \{\pm\infty\}$ for any $i \in \mathbb{N}$ with $1 \le i \le n$ where $\gamma_i \ne 0$. The following implications are therefore justified:

$$\text{`}\sum_{i=1}^{n} \gamma_i x_i > 0\text{'} \in \mathrm{tp}(\bar{x}) \iff \Gamma \models \sum_{i=1}^{n} \gamma_i x_i > 0, \tag{8.6}$$

$$\iff \mathrm{st}\left(\frac{\sum_{i=1}^{n} \gamma_i x_i}{x_j}\right) > 0, \tag{8.7}$$

$$\iff \sum_{i=1}^{n} \gamma_i \,\mathrm{st}\left(\frac{x_i}{x_j}\right) > 0, \tag{8.8}$$

$$\iff \sum_{i=1}^{n} \gamma_i \,\mathrm{st}\left(\frac{y_i}{y_j}\right) > 0, \tag{8.9}$$

$$\iff \text{`}\sum_{i=1}^{n} \gamma_i y_i > 0\text{'} \in \mathrm{tp}(\bar{y})\,. \tag{8.10}$$

It is by strong independence and lemma 8.2.5 that we ensure $\mathrm{v}(\sum_{i=1}^{n} \gamma_i x_i) = \mathrm{v}(x_j)$ and hence that the implications (8.6) $\Rightarrow$ (8.7) and (8.10) $\Rightarrow$ (8.9) hold.

It is clear then that $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ as required. □

Definition 8.2.12. Let $\gamma_1, \gamma_2 \in \Gamma$. We say that γ_1 is **close to** γ_2 if

$$\mathrm{st}\left(\frac{\gamma_1}{\gamma_2}\right) = 1 \quad \text{or} \quad \gamma_1 = \gamma_2 = 0.$$

We denote this property by writing $\gamma_1 \frown \gamma_2$ and extend it to $\widetilde{\Gamma}$ in the obvious way.

Lemma 8.2.13. Suppose $\gamma, \gamma_1, \gamma_2 \in \widetilde{\Gamma}$ are such that

$$\mathrm{st}\left(\frac{\gamma_1}{\gamma}\right) = \mathrm{st}\left(\frac{\gamma_2}{\gamma}\right) \notin \{0, \pm\infty\}.$$

Then $\gamma_1 \frown \gamma_2$.

Proof. Using lemma 8.1.5 the result follows trivially:-

$$\mathrm{st}\left(\frac{\gamma_1}{\gamma_2}\right) = \mathrm{st}\left(\frac{\gamma_1}{\gamma}\right) \cdot \mathrm{st}\left(\frac{\gamma}{\gamma_2}\right) = \mathrm{st}\left(\frac{\gamma_1}{\gamma}\right) \cdot \mathrm{st}\left(\frac{\gamma}{\gamma_1}\right) = 1.$$

□

It is clear from the definitions that, as with value, closeness is an equivalence relation and it can be extended to elements of $\widetilde{\Gamma}$ in an obvious way.

Chapter 9

Recursive Saturation

9.1 Notation

In this and the remaining chapters we will begin to concentrate on a particular subclass of models of Presburger arithmetic and their automorphisms. Because of this there are a number of recurring concepts which it will be useful to provide notation for. We will also simplify our terminology somewhat by referring to the automorphism group $\mathrm{Aut}(\Gamma)$ as just G and in keeping with earlier notation we set $\widetilde{G} = \mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})$.

We also wish to delineate between two types of automorphisms: those which 'preserve' values and those which 'defy' values. We will give these precise definitions.

Definition 9.1.1. For $g \in G$, we say that g is a **value-preserving automorphism** if $\mathrm{v}(\gamma g) = \mathrm{v}(\gamma)$ for all $\gamma \in \Gamma$. For $g \in G$, we say that g is a **value-defying automorphism** if there exists some $\gamma \in \Gamma$ for which $\mathrm{v}(\gamma g) \neq \mathrm{v}(\gamma)$.

Note that the two concepts are mutually exclusive and exhaustive. It will also be useful to give the set of value-preserving automorphisms a name and to this end we set

$$G_{\mathrm{v}} = \{\, g \in G : \forall\, \gamma \in \Gamma \mathrm{v}(\gamma g) = \mathrm{v}(\gamma)\}.$$

We then have that $G \setminus G_{\mathrm{v}}$ is the set of value-defying automorphisms. The set G_{v} is particularly important in our analysis, being a subgroup possessing many useful and important properties. These are discussed throughout, but are addressed specifically in section 11.3.

9.2 Preamble

A model M is said to be recursively saturated if and only if every recursive type over M is realized in M. Many results can be derived by concentrating exclusively on recursively saturated models, and this is no less true when considering models of Presburger arithmetic in particular. To illustrate this we produce the following lemma, which is a restricted version of a lemma to be found in Kaye [29]. The proof is also taken from here.

Lemma 9.2.1. Let M be a recursively saturated $\mathscr{L}$ structure where $\mathscr{L}$ is a recursive language. Then if $\bar{a}, \bar{b} \in M$ are tuples of the same finite length, with $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ and $c \in M$ is arbitrary, then there exists $d \in M$ such that $\mathrm{tp}(\bar{a}, c) = \mathrm{tp}(\bar{b}, d)$.

Proof. Let d be chosen to satisfy the recursive type

$$p(x) = \{\theta(\bar{a}, c) \rightarrow \theta(\bar{b}, x) : \theta \text{ an } \mathscr{L}\text{-formula}\}.$$

This is clearly recursive and is a type since $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, so if $M \models \bigwedge_{i=1}^{n} \theta_i(\bar{a}, c)$ then $M \models \exists\, x \bigwedge_{i=1}^{n} \theta_i(\bar{a}, x)$, hence $M \models \exists\, x \bigwedge_{i=1}^{n} \theta_i(\bar{b}, x)$. □

As a corollary to this we clearly have the following:

Corollary 9.2.2. If Γ is a countable, recursively saturated model of Presburger arithmetic then Γ is homogeneous.

Proof. Use lemma 9.2.1 to produce a simple back-and-forth construction. □

However, when considering countable models of Presburger arithmetic, the property of recursive saturation is very strong; stronger in fact than is often necessary when you are mainly interested in automorphisms rather than the more general case of homomorphisms. The following lemma illustrates the extent of the restriction on models which recursive saturation imposes:

Lemma 9.2.3. Suppose Γ is a countable, recursively saturated model of Presburger arithmetic. Then $\mathrm{Res}(\Gamma)$ contains every recursive element of $\widehat{\mathbb{Z}}$ and $\mathrm{stQ}(\Gamma)$ contains every recursive element of $\mathbb{R}$ and the elements $\pm\infty$.

Proof. If $\bar{z} \in \widehat{\mathbb{Z}}$ is recursive then the following type is clearly recursive:

$$\{\, x \equiv z_n \pmod{n} : n \in \mathbb{N}\} \quad \text{where } \bar{z} = \{z_1, z_2, \ldots\}.$$

By saturation this type is therefore realized by some element d where $\varrho(d) = \bar{z}$.

For the second part, let a be a recursive element of $\mathbb{R}$ or one of $\pm\infty$. Then for d_1, d_2 satisfying the recursive type

$$\left\{ sx_1 - rx_2 > 0 : \frac{r}{s} < a \right\} \cup \left\{ rx_2 - sx_1 > 0 : \frac{r}{s} > a \right\}$$

we see that $\operatorname{st}\left(\frac{d_1}{d_2}\right) = a$. □

We wish to utilise a weaker property than recursive saturation, whilst still retaining its beneficial aspects. In the next section we define such a notion.

9.3 Pseudo-recursive saturation

In this section we will define a new class of models which are a superset of the recursively saturated models. These we will call the pseudo-recursively saturated models, and the original idea behind them can be found in Harnik [27] where they are used but not explicitly defined. The first explicit definition can be found in Kaye [31], where the idea is developed more fully. However, in both cases the notion is used as a means to an end; a way of describing a set of models which satisfy the properties required in order to ensure automorphisms are both easily constructible and abundant (by which we mean that such models are homogeneous, a proof of which can be found in Kaye [31] and also below).

We will approach the situation from a slightly different perspective, concentrating on the close relationship between homogeneous and pseudo-recursively saturated models. We will therefore look at some of the properties of homogeneous models and see how the pseudo-recursively saturated models can be derived from these. In this way we hope that the motivation for concentrating on pseudo-recursively saturated models will become apparent over and above their expedient nature.

In order to do this we provide a number of lemmas which we will need in order to prove the main results.

Lemma 9.3.1. Let Γ be a model of Presburger arithmetic with elements $\gamma_1 < \gamma_2 \in \Gamma$ such that $\varrho(\gamma_1) = \varrho(\gamma_2)$ and $\operatorname{st}\left(\frac{\gamma_1}{\gamma_2}\right) \neq 1$. Then $\mathrm{v}(\gamma_2)$ contains an instance of every residue in $\mathrm{Res}(\{\, x : \mathrm{v}(x) \leq \mathrm{v}(\gamma_2) \})$.

Proof. Since $\operatorname{st}\left(\frac{\gamma_1}{\gamma_2}\right) \neq 1$ we know that $\mathrm{v}(\gamma_2 - \gamma_1) = \mathrm{v}(\gamma_2)$ and $\varrho(\gamma_2 - \gamma_1) = 0$. Choose any x with $\mathrm{v}(x) < \mathrm{v}(\gamma_2)$. Then $\mathrm{v}(\gamma_2 - \gamma_1 + x) = \mathrm{v}(\gamma_2)$ and $\varrho(\gamma_2 - \gamma_1 + x) = \varrho(x)$ as required. □

Lemma 9.3.2. Let Γ be a 1-homogeneous model of Presburger arithmetic with no smallest value and elements $\gamma_1 < \gamma_2 \in \Gamma$ such that $\varrho(\gamma_1) = \varrho(\gamma_2)$ and $\operatorname{st}\left(\frac{\gamma_1}{\gamma_2}\right) \neq 1$. Then for every $x_1, x_2 \in \Gamma$ with $\mathrm{v}(x_1), \mathrm{v}(x_2) \leq \mathrm{v}(\gamma_2)$ and $x_1 + \mathbb{Z} < x_2$ and every $r \in \operatorname{Res}(\{\, x : \mathrm{v}(x) \leq \mathrm{v}(\gamma_2)\})$ there is some $x_1 < x_3 < x_2$ such that $\varrho(x_3) = r$.

Proof. We know by lemma 9.3.1 that $\mathrm{v}(\gamma_2)$ contains an instance of every residue in $\operatorname{Res}(\{\, x : \mathrm{v}(x) \leq \mathrm{v}(\gamma_2)\})$. Now take any valuation class $\mathrm{v}(\gamma_3) < \mathrm{v}(\gamma_2)$ for some $\gamma_3 \in \Gamma$. By the above reasoning, there is some element γ_4 with $\mathrm{v}(\gamma_4) = \mathrm{v}(\gamma_2)$ and $\varrho(\gamma_4) = \varrho(\gamma_3)$. But then by 1-homogeneity there exists an automorphism $g\colon \Gamma \to \Gamma$ such that $g\colon \gamma_4 \mapsto \gamma_3$. We know that $\operatorname{st}\left(\frac{\gamma_1}{\gamma_2}\right) \neq 1$, so $\mathrm{v}(\gamma_2 - \gamma_1) = \mathrm{v}(\gamma_2)$ and by choosing some $x \in \Gamma$ with $\mathrm{v}(x) < \mathrm{v}(\gamma_2)$ we see that $\mathrm{v}((\gamma_2 - \gamma_1 + x)g) = \mathrm{v}((\gamma_2 - \gamma_1)g) = \mathrm{v}(\gamma_2 g) = \mathrm{v}(\gamma_4 g) = \mathrm{v}(\gamma_3)$. Thus every valuation class $\mathrm{v}(\gamma_3) \leq \mathrm{v}(\gamma_2)$ contains an instance of each residue in $\operatorname{Res}(\{\, x : \mathrm{v}(x) \leq \mathrm{v}(\gamma_2)\})$.

Now let x_1, x_2 be such that $\mathrm{v}(x_1), \mathrm{v}(x_2) \leq \mathrm{v}(\gamma_2)$ and $x_1 + \mathbb{Z} < x_2$. Thus $\mathrm{v}(x_2 - x_1) > 0$. Choosing some $r \in \operatorname{Res}(\{\, x : \mathrm{v}(x) \leq \mathrm{v}(\gamma_2)\})$ we can use the previous arguments to find some $x_4 \in \Gamma$ with $0 < \mathrm{v}(x_4) < \mathrm{v}(x_2 - x_1)$ and $\varrho(x_4) = r - \varrho(x_1)$. But then setting $x_3 = x_1 + x_4$ we see that $\varrho(x_3) = \varrho(x_1 + x_4) = \varrho(x_1) + r - \varrho(x_1) = r$ and $x_1 < x_3 < x_2$ as required. □

Lemma 9.3.3. Suppose Γ is a 1-homogeneous model of Presburger arithmetic with values forming a dense linear order and such that $\varrho^{-1}(r)/\mathbb{Z}$ is dense in $\Gamma/\mathbb{Z}$ for all $r \in \operatorname{Res}(\Gamma)$. Then for every non-standard $x, y, z \in \Gamma$ there exists some $w \in \Gamma$ such that $\operatorname{st}\left(\frac{w}{z}\right) = \operatorname{st}\left(\frac{x}{y}\right)$.

Proof. Let $x, y, z \in \Gamma$ be non-standard. We want to find some $w \in \Gamma$ such that

$$\operatorname{st}\left(\frac{x}{y}\right) = \operatorname{st}\left(\frac{w}{z}\right).$$

We may assume that $\operatorname{st}\left(\frac{x}{y}\right) \notin \{0, \pm\infty\}$, as these cases follow from the fact that the values form a dense linear order. By the denseness of residues we can find some z' with $\varrho(z') = \varrho(y)$ and $\operatorname{st}\left(\frac{z'}{z}\right) = 1$. But then by 1-homogeneity there is some automorphism $g\colon \Gamma \to \Gamma$ such that $g\colon y \mapsto z'$. Set $w = xg$. Then

$$\operatorname{st}\left(\frac{w}{z}\right) = \operatorname{st}\left(\frac{w}{z'}\right) = \operatorname{st}\left(\frac{xg}{yg}\right) = \operatorname{st}\left(\frac{x}{y}\right)$$

which is the result required. □

The next theorem provides a link between the homogeneous models and the pseudo-recursively saturated models. The convex submodel described in the fourth item below satisfies precisely those properties which we will use to define pseudo-recursive saturation, the strict definition of which is given directly afterwards. In the interesting case of a homogeneous model with value-defying automorphism, then, we see that pseudo-recursive saturation plays in integral part.

Notation. We say that Γ is n-homogeneous if for any n-tuples $\bar{a}$ and $\bar{b}$ with the same type, there exists an automorphism mapping one to the other.

It's worth mentioning that although in the next theorem we take Γ to be 2-homogeneous, this can be reduced to 1-homogeneity if we add the requirement that the values of Γ should form a dense linear order in item three.

Theorem 9.3.4. Suppose that Γ is a 2-homogeneous model of Presburger arithmetic. Then the following are equivalent:

1. Γ has no smallest non-standard value, and there exists a non-trivial automorphism $g\colon \Gamma \to \Gamma$;
2. there is some element $x \in \Gamma$ such that $\varrho(x) = 0$ and there are non-standard elements with value less than that of x;
3. there is some value-defying automorphism $h\colon \Gamma \to \Gamma$;
4. Γ contains a convex submodel Γ' with values forming a dense linear order, with $\widetilde{\varrho}^{-1}(r)$ dense in $\widetilde{\Gamma}'$ for all $r \in \mathrm{Res}(\widetilde{\Gamma}')$ and so that for all non-standard $x, y, z \in \Gamma'$ there exists some $w \in \Gamma'$ such that $\mathrm{st}\left(\frac{w}{z}\right) = \mathrm{st}\left(\frac{x}{y}\right)$.

Proof. (1) $\implies$ (2) Since g is non-trivial there exists some $x \in \Gamma$ such that $x \neq xg$. Therefore $\varrho(xg) = \varrho(x)$ and hence we also note that $xg \notin x + \mathbb{Z}$. Suppose without loss of generality that $x > xg$. Then $\mathrm{v}(x - xg) > 0$ and $\varrho(x - xg) = 0$. Since Γ has no smallest non-standard value, it follows that there are non-standard elements with value less than that of $x - xg$.

(2) $\implies$ (3) Choose some element x' with $0 < \mathrm{v}(x') < \mathrm{v}(x)$. We can do this since there are values smaller than that of x. Then $\varrho(x') = \varrho(x + x')$ so by 1-homogeneity

there is an automorphism $h\colon \Gamma \to \Gamma$ which maps $h\colon x' \mapsto x + x'$. The map h therefore defies values, as required.

(3) $\implies$ (4) Since h is value-defying we know that there must be some $\gamma_1, \gamma_2 \in \Gamma$ so that $\mathrm{v}(\gamma_1) < \mathrm{v}(\gamma_2)$ and $\varrho(\gamma_1) = \varrho(\gamma_2)$. Lemma 9.3.1 then tells is that $\mathrm{v}(\gamma_2)$ contains an instance of every residue in $\mathrm{Res}(\{\, x : \mathrm{v}(x) \leq \mathrm{v}(\gamma_2)\})$. We claim that Γ has no smallest value. So suppose that there is a smallest value containing some element γ_3, say. We know that $\mathrm{v}(\gamma_3) < \mathrm{v}(\gamma_2)$ because $\mathrm{v}(\gamma_1) < \mathrm{v}(\gamma_2)$ and hence $\mathrm{v}(\gamma_2)$ cannot be the smallest value. But then by lemma 9.3.1 there is some γ_4 with $\mathrm{v}(\gamma_4) = \mathrm{v}(\gamma_2)$ and $\varrho(\gamma_4) = \varrho(\gamma_3)$, and so by 1-homogeneity we can find some automorphism $g\colon \Gamma \to \Gamma$ which maps $\gamma_4 \mapsto \gamma_3$. Since $\mathrm{v}(\gamma_1) < \mathrm{v}(\gamma_2)$ we then know that $\mathrm{v}(\gamma_1 g) < \mathrm{v}(\gamma_2 g) = \mathrm{v}(\gamma_4 g) = \mathrm{v}(\gamma_3)$, contradicting the hypothesis that $\mathrm{v}(\gamma_3)$ is the smallest value. Γ can therefore have no smallest value.

It now follows by 9.3.2 that for every $x_1, x_2 \in \Gamma$ with $\mathrm{v}(x_1), \mathrm{v}(x_2) \leq \mathrm{v}(\gamma_2)$ and $x_1 + \mathbb{Z} < x_2$ and every $r \in \mathrm{Res}(\{\, x : \mathrm{v}(x) \leq \mathrm{v}(\gamma_2)\})$ there is some $x_1 < x_3 < x_2$ such that $\varrho(x_3) = r$.

We also claim that for every $x_1, x_2 \in \Gamma$ with $\mathrm{v}(x_1) < \mathrm{v}(x_2) \leq \mathrm{v}(\gamma_2)$ there exists some x_3 with $\mathrm{v}(x_1) < \mathrm{v}(x_3) < \mathrm{v}(x_2)$. But this follows immediately from 2-homogeneity, for we can find some x_4 with $\mathrm{v}(x_4) < \mathrm{v}(x_1)$ and $\varrho(x_4) = \varrho(x_1)$ and hence an automorphism $h'\colon \Gamma \to \Gamma$ which maps:

$$h'\colon \begin{array}{ccc} x_4 & \mapsto & x_1 \quad ; \\ x_2 & \mapsto & x_2 \quad . \end{array}$$

Setting $x_3 = x_1 g$ it is then clear that $\mathrm{v}(x_1) < \mathrm{v}(x_3) < \mathrm{v}(x_2)$ as required. Note that this is the only time 2-homogeneity is needed; only 1-homogeneity is needed elsewhere.

Now consider the set of values

$$V' = \{\, \mathrm{v}(x) \in V : \exists\, g \in G, \gamma_1, \gamma_2 \in \Gamma, \mathrm{v}(\gamma_1) < \mathrm{v}(\gamma_2), \mathrm{v}(\gamma_2) = \mathrm{v}(x) \text{ and } \gamma_1 g = \gamma_2\}.$$

In other words, the set of values which are moved by some automorphism of Γ. We claim that the set of elements

$$\Gamma' = \{\, \gamma \in \Gamma : \mathrm{v}(\gamma) \in V' \cup \{0\}\}$$

is a convex submodel of Γ of the required sort.

First of all, we note that $V' \neq \emptyset$ by our assumption that there is a value-defying automorphism of Γ. Also, V' forms a dense linear order with no maximum point. For suppose $\mathrm{v}(\gamma_1) < \mathrm{v}(\gamma_2) \in V'$. By lemma 9.3.2, if $\mathrm{v}(\gamma_2) \in V'$ and $\gamma_1 < \gamma_2$ then $\mathrm{v}(\gamma_1) \in V'$ and the values of V' are dense as just shown. Hence there is some $\mathrm{v}(\gamma_3) \in V'$ with $\mathrm{v}(\gamma_1) < \mathrm{v}(\gamma_2) < \mathrm{v}(\gamma_3)$. Now suppose that V' had a max point, $\mathrm{v}(x)$, say. Then by the definition of V' there is some γ_1, γ_2 with $\mathrm{v}(\gamma_2) = \mathrm{v}(x)$ and some automorphism $g\colon \Gamma \to \Gamma$ such that $\gamma_1 g = \gamma_2$. Clearly we may suppose that $\mathrm{v}(\gamma_1) < \mathrm{v}(\gamma_2)$ and so $\mathrm{v}(\gamma_2) < \mathrm{v}(\gamma_2 g)$. But then $\mathrm{v}(\gamma_2 g) \in V'$ contradicting the assumption that $\mathrm{v}(\gamma_2)$ is maximum in V'.

It follows by lemma 9.3.2 that $\widetilde{\varrho}^{-1}(r)$ is dense in $\widetilde{\Gamma}'$ for all $r \in \mathrm{Res}(\widetilde{\Gamma}')\Gamma'$. Finally it follows by lemma 9.3.3 that for every non-standard $x, y, z \in \Gamma'$ there exists some $w \in \Gamma'$ such that $\mathrm{st}\left(\frac{w}{z}\right) = \mathrm{st}\left(\frac{x}{y}\right)$.

Clearly the model Γ' satisfies the requirements.

(4) $\implies$ (1) This follows immediately from the fact that Γ' is convex and contains no smallest value. Denseness of residues ensures that there are plenty of non-trivial automorphisms which act on the model. □

We shall discover later on that the convex submodel Γ' in the fourth item from theorem 9.3.4 is uniquely maximal with the given properties. This results from its definition as the largest submodel admitting automorphisms between distinct values and will become apparent from corollary 9.3.10. In fact this fourth item gives rise to the definition of pseudo-recursive saturation which we will now present. This definition will be used extensively throughout the remainder of this thesis.

Definition 9.3.5. A Presburger group Γ is **pseudo-recursively saturated** if $\Gamma \ncong \mathbb{Z}$ and

1. for $\widetilde{\varrho}\colon \Gamma/\mathbb{Z} \to \widehat{\mathbb{Z}}/\mathbb{Z}$ and each $\mathbb{Z}+r \in \mathrm{Im}(\widetilde{\varrho})$, the inverse image $\widetilde{\varrho}^{-1}(\mathbb{Z}+r)$ is dense in $\Gamma/\mathbb{Z}$ (in the sense of $<$);

2. for $x, y, z \in \Gamma$ with $z \notin \mathbb{Z}$, there is some $w \notin \mathbb{Z}$ for which
$$\mathrm{st}\left(\frac{w}{z}\right) = \mathrm{st}\left(\frac{x}{y}\right);$$

3. the set of values $\{\, \mathrm{v}(x) : x \in \Gamma/\mathbb{Z}\}$ is a dense linear order with respect to $<$ having least point 0 and no greatest point.

Part 2 of the definition above does not preclude the possibility that $\mathrm{st}\left(\frac{x}{y}\right) \in \{0, \pm\infty\}$ since this in fact follows immediately from part 1. With reference to the definition of standard parts (8.1.4) we note that we may also re-write part 2 as:

2. for $x, y, z \in \Gamma/\mathbb{Z}$ with $z \notin \mathbb{Z} + 0$, there is some $w \notin \mathbb{Z} + 0$ for which

$$\mathrm{st}\left(\frac{w}{z}\right) = \mathrm{st}\left(\frac{x}{y}\right).$$

This form is perhaps more in keeping with parts 1 and 3.

Proposition 9.3.6. Any recursively saturated model of Presburger arithmetic is pseudo-recursively saturated.

Proof. The proof of this is straightforward, and can be found as propositions 3, 5 and 7 in Harnik [27]. □

Pseudo-recursive saturation was originally presented (in Harnik [27] and Kaye [31] as described earlier) to facilitate the construction of automorphisms and so—despite being considerably weaker than recursive saturation—it is still surprisingly useful. To demonstrate this we shall emulate the result of corollary 9.2.2 for pseudo-recursively saturated models and to do this we use the extension of the forward direction of lemma 8.2.11 mentioned previously. This extension—given in the next theorem—is of key importance in exhibiting the strength of pseudo-recursive saturation, especially with respect to the construction of automorphisms. Combined with lemma 8.2.11, it also gives a good indication of the close connection between types, standard parts and residues which exists in Presburger arithmetic, a connection made possible by the 'limited descriptive power' which quantifier elimination within the language imposes.

For these theorems involving the construction of automorphisms, the following lemma turns out to be useful.

Lemma 9.3.7. Let Γ be pseudo-recursively saturated with $\gamma_1 \in \widetilde{\Gamma}\backslash\{0\}$ and $r \in \mathrm{Res}(\widetilde{\Gamma})$. Then there exists some $\gamma_2 \in \widetilde{\Gamma}$ such that $\gamma_1 \frown \gamma_2$ and $\widetilde{\varrho}(\gamma_2) = r$.

Proof. The element γ_1 is non-zero, so by p.r.s.(3) there exists some positive element $c \in \widetilde{\Gamma} \setminus \{0\}$ with $\mathrm{v}(c) < \mathrm{v}(\gamma_1)$. But then $\gamma_1 < \gamma_1 + c$ and so by p.r.s.(1) there exists some element γ_2 with $\gamma_1 < \gamma_2 < \gamma_1 + c$ and $\widetilde{\varrho}(\gamma_2) = r$. It remains to show that $\gamma_1 \frown \gamma_2$. Let $\gamma_2 - \gamma_1 = d$. Then it is clear that $d < c$ and hence $\mathrm{v}(d) < \mathrm{v}(\gamma_1)$. It follows that

$$\mathrm{st}\left(\frac{\gamma_2}{\gamma_1}\right) = \mathrm{st}\left(\frac{\gamma_1 + d}{\gamma_1}\right) = \mathrm{st}\left(\frac{\gamma_1}{\gamma_1}\right) + \mathrm{st}\left(\frac{d}{\gamma_1}\right) = 1 + 0,$$

which provides us with the required result. □

In the next theorem we abuse notation somewhat by stating that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ for $\bar{a}, \bar{b} \in \widetilde{\Gamma}$. In essence this is intended to mean that there exist $\bar{a}', \bar{b}' \in \Gamma$ such that $\mathrm{tp}(\bar{a}') = \mathrm{tp}(\bar{b}')$ and $\bar{a} = \bar{a}'/\mathbb{Z}, \bar{b} = \bar{b}'/\mathbb{Z}$.

Theorem 9.3.8. Let Γ be countable pseudo-recursively saturated and suppose that $(a_1, \ldots, a_n) = \bar{a} \in \widetilde{\Gamma}$ and $(b_1, \ldots, b_n) = \bar{b} \in \widetilde{\Gamma}$ are such that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. Then there exist strongly independent sets $\{a'_1, \ldots, a'_{n'}\}$ and $\{b'_1, \ldots, b'_{n'}\}$ where $n' \leq n$ which satisfy the following:

1. $\langle a_1, \ldots, a_n \rangle = \langle a'_1, \ldots, a'_{n'} \rangle$ and $\langle b_1, \ldots, b_n \rangle = \langle b'_1, \ldots, b'_{n'} \rangle$;
2. $a_i = q_1 a'_1 + \cdots + q_{n'} a'_{n'}$ if and only if $b_i = q_1 b'_1 + \cdots + q_{n'} b'_{n'}$ where $q_1, \ldots, q_{n'} \in \mathbb{Q}$;
3. $\widetilde{\varrho}(a'_i) = \widetilde{\varrho}(b'_i)$ for $i \in \mathbb{N}$ with $1 \leq i \leq n'$;
4. $\mathrm{st}\left(\frac{a'_i}{a'_j}\right) = \mathrm{st}\left(\frac{b'_i}{b'_j}\right)$ for $i, j \in \mathbb{N}$ with $1 \leq i \leq j \leq n'$.

Proof. We prove this inductively.

The base case when $n = 1$ is trivial.

For the inductive step, suppose we know it is true for some $n = m-1$. We therefore have

$$\langle a_1, \ldots, a_{m-1} \rangle = \left\langle a'_1, \ldots, a'_{m'-1} \right\rangle \quad \text{and} \quad \langle b_1, \ldots, b_{m-1} \rangle = \left\langle b'_1, \ldots, b'_{m'-1} \right\rangle$$

where $m' \leq m$. We hope to show that it is also true for $n = m$. By the Exchange Lemma (lemma 8.2.10) we can find $a'_{m'} \in \left\langle a'_1, \ldots, a'_{m'-1}, a_m \right\rangle$ such that $a_m \in \left\langle a'_1, \ldots, a'_{m'-1}, a'_{m'} \right\rangle$ with $\{a'_1, \ldots .a'_{m'-1}, a'_{m'}\}$ strongly independent. Now

$$a'_{m'} = q_1 a'_1 + \cdots + q_{m'-1} a'_{m'-1} + q_{m'} a_m \tag{9.1}$$

for some $q_1, \ldots, q_{m'} \in \mathbb{Q}$ and so we set

$$b'_{m'} = q_1 b'_1 + \cdots + q_{m'-1} b'_{m'-1} + q_{m'} b_m. \tag{9.2}$$

We claim that this will suffice, with the remainder of the proof devoted to verifying the fact.

The criteria 2 and 3 above follow trivially from the inductive hypothesis and the equations 9.1 and 9.2 above.

For criterion 1, we first notice that $a_1, \ldots, a_{m-1} \in \left\langle a'_1, \ldots a'_{m'-1} \right\rangle$ by the inductive hypothesis and that $a_m \in \langle a'_1, \ldots, a'_{m'} \rangle$ by 9.1 above. It then follows trivially that $\langle a_1, \dots, a_m \rangle \subseteq \langle a'_1, \dots, a'_{m'} \rangle$. For the reverse direction, we know that $a'_{m'} \in \left\langle a'_1, \dots, a'_{m'-1}, a_m \right\rangle$ and by our inductive hypothesis we know that $\left\langle a'_1, \dots, a'_{m'-1} \right\rangle = \langle a_1, \dots, a_{m-1} \rangle$. Hence $a'_{m'} \in \langle a_1, \dots a_{m-1}, a_m \rangle$ as clearly $a'_1, \dots a'_{m'-1}$ are. It follows that $\langle a_1, \dots, a_m \rangle \supseteq \langle a'_1, \dots, a'_{m'} \rangle$ as required.

In order to show that $\langle b_1, \dots, b_m \rangle = \langle b'_1, \dots, b'_{m'} \rangle$ we first note that by our inductive hypothesis $\left\langle b'_1, \dots, b'_{m'-1} \right\rangle = \langle b_1, \dots, b_{m-1} \rangle$. The fact that $b'_{m'} \in \langle b_1, \dots, b_m \rangle$ then follows from this and equation 9.2. The reverse direction follows as a result of criterion 2 holding and the fact shown above that $\langle a_1, \dots, a_m \rangle \subseteq \langle a'_1, \dots, a'_{m'} \rangle$. For suppose $i \in \mathbb{N}$ with $1 \leq i \leq m$. Then $a_i = q_1 a'_1 + \dots + q_{m'} a'_{m'}$ implies that $b_i = q_1 b'_1 + \dots + q_{m'} b'_{m'}$ and hence $b_i \in \langle b'_1, \dots, b'_{m'} \rangle$. We may conclude from this that $\langle b_1, \dots b_m \rangle \subseteq \langle b'_1, \dots, b'_{m'} \rangle$.

If we are able to show that criterion 4 holds, then the strong independence of the set $\{b'_1, \dots, b'_{m'-1}, b'_{m'}\}$ will follow from the fact that $\{a'_1, \dots, a'_{m'-1}, a_m\}$ is a strongly independent set. So for criterion 4, let $i \in \mathbb{N}$ with $1 \leq i \leq m' - 1$ be such that $\mathrm{v}(b'_i) = \max\{\mathrm{v}(b_j) : 1 \leq j \leq m-1, q_j \neq 0\}$. Then

$$\mathrm{st}\left(\frac{q_j b'_j}{b'_i}\right) \notin \{\ \pm\infty\} \quad \text{for } j \in \mathbb{N} \text{ with } 1 \leq j \leq m' - 1,$$

so the following must hold:

$$\begin{aligned} \mathrm{st}\left(\frac{b'_{m'}}{b'_i}\right) &= \mathrm{st}\left(\frac{q_1 b'_1 + \dots + q_{m'-1} b'_{m'-1} + q_{m'} b_m}{b'_i}\right) \\ &= \mathrm{st}\left(\frac{q_1 b'_1}{b'_i}\right) + \dots + \mathrm{st}\left(\frac{q_{m'-1} b'_{m'-1}}{b'_i}\right) + \mathrm{st}\left(\frac{q_{m'} b_m}{b'_i}\right) \\ &= \mathrm{st}\left(\frac{q_1 a'_1}{a'_i}\right) + \dots + \mathrm{st}\left(\frac{q_{m'-1} a'_{m'-1}}{a'_i}\right) + \mathrm{st}\left(\frac{q_{m'} b_m}{b'_i}\right), \end{aligned}$$

where the last equality holds by our inductive assumptions. We must show that $\mathrm{st}\left(\frac{b_m}{b'_i}\right) = \mathrm{st}\left(\frac{a_m}{a'_i}\right)$.

By the criteria 1 and 2 we can see that

$$a'_i = q'_1 a_1 + \dots + q'_i a_i \quad \text{and} \quad b'_i = q'_1 b_1 + \dots + q'_i b_i$$

for some $q'_1, \dots, q'_i \in \mathbb{Q}$. But we also know that $\mathrm{tp}(a_1, \dots, a_n) = \mathrm{tp}(b_1, \dots, b_n)$ and so we may deduce that $\mathrm{tp}(a'_i, a_m) = \mathrm{tp}(b'_i, b_m)$. It follows that $\mathrm{st}\left(\frac{b_m}{b'_i}\right) = \mathrm{st}\left(\frac{a_m}{a'_i}\right)$ by lemma 8.2.11. Attention should be drawn to the fact that although a requirement of strong independence is suggested in the statement of lemma 8.2.11, it is not actually necessary for the direction in which we need it (see the note in the proof on page 68).

Finally then we see that

$$\begin{aligned}\operatorname{st}\left(\frac{b'_{m'}}{b'_i}\right) &= \operatorname{st}\left(\frac{q_1 a'_1}{a'_i}\right) + \cdots + \operatorname{st}\left(\frac{q_{m'-1} a'_{m'-1}}{a'_i}\right) + \operatorname{st}\left(\frac{q_{m'} a_m}{a'_i}\right) \\ &= \operatorname{st}\left(\frac{q_1 a_1 + \cdots + q_{m'-1} a'_{m'-1} + q_{m'} a_m}{a'_i}\right) \\ &= \operatorname{st}\left(\frac{a'_{m'}}{a'_i}\right).\end{aligned}$$

The criterion 4 then follows from this, since for any $j \in \mathbb{N}$ with $1 \leq j \leq m'$,

$$\begin{aligned}\operatorname{st}\left(\frac{b'_{m'}}{b'_j}\right) &= \operatorname{st}\left(\frac{b'_{m'}}{b'_i}\right) \cdot \operatorname{st}\left(\frac{b'_i}{b'_j}\right) \\ &= \operatorname{st}\left(\frac{a'_{m'}}{a'_i}\right) \cdot \operatorname{st}\left(\frac{a'_i}{a'_j}\right) \\ &= \operatorname{st}\left(\frac{a'_{m'}}{a'_j}\right).\end{aligned}$$

□

We may now provide the following two results as the P.R.S. counterparts to lemma 9.2.1 and corollary 9.2.2 respectively.

Lemma 9.3.9. Let Γ be a countable, pseudo-recursively saturated model of Presburger arithmetic. Then if $\bar{x}, \bar{y} \in \widetilde{\Gamma} \backslash \{0\}$ are tuples of the same finite length, with $\operatorname{tp}(\bar{x}) = \operatorname{tp}(\bar{y})$ and $z \in \widetilde{\Gamma}$ arbitrary, then there exists $w \in \widetilde{\Gamma}$ such that $\operatorname{tp}(\bar{x}, z) = \operatorname{tp}(\bar{y}, w)$.

Proof. Write

$$\bar{x} = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad \bar{y} = (y_1, y_2, \ldots, y_n)$$

for some $n \in \mathbb{N}$. By theorem 9.3.8 we can therefore find strongly independent sets

$$\bar{x}' = (x'_1, x'_2, \ldots, x'_{n'}) \quad \text{and} \quad \bar{y}' = (y'_1, y'_2, \ldots, y'_{n'})$$

and satisfying the criteria 1–4 as given in that theorem. Now suppose that $z \in \langle x'_1, x'_2, \ldots, x'_{n'} \rangle$. Then $z = q_1 x'_1 + \cdots q_{n'} x'_{n'}$ for some $q_1, \ldots, q_{n'} \in \mathbb{Q}$. In this case we set $w = q_1 y'_1 + \cdots q_{n'} y'_{n'}$. Since $\widetilde{\varrho}(x_i) = \widetilde{\varrho}(y_i)$ for all i it follows that $\widetilde{\varrho}(z) = \widetilde{\varrho}(w)$ as required. In this first case we set $m = n'$.

Suppose on the other hand that $z \notin \langle x'_1, x'_2, \ldots, x'_{n'} \rangle$. Then by the Exchange Lemma (lemma 8.2.10) we can find $x'_{n'+1}$ so that $\{x'_1, \ldots, x'_{n'}, x'_{n'+1}\}$ is a strongly independent set with z contained in its span.

Now suppose $\mathrm{v}(x'_{n'+1}) = \mathrm{v}(x'_i)$ for some i. Then $\mathrm{st}\left(\frac{x'_{n'+1}}{x'_i}\right) = r$ for some $r \in \mathbb{R}\setminus\{0\}$. So by p.r.s.(2) we can find $w'' \in \widetilde{\Gamma}$ such that $\mathrm{st}\left(\frac{w''}{y'_i}\right) = r$. If $\mathrm{v}(x'_{n'+1}) \neq \mathrm{v}(x'_i)$ for any i then choose any w'' so that $\mathrm{v}(w'') \neq \mathrm{v}(y'_i)$ for any i. We can do this by p.r.s.(3). In both cases it is clear that $\mathrm{st}\left(\frac{x'_{n'+1}}{x'_i}\right) = \mathrm{st}\left(\frac{w''}{y'_i}\right)$ for all i.

Now let $c \in \widetilde{\Gamma} \setminus \{0\}$ be such that $\mathrm{v}(c) < \min\{\mathrm{v}(y'_1), \ldots, \mathrm{v}(y'_{n'})\}$. Again this is possible since $y'_i \notin \mathbb{Z}$ for any i and by p.r.s.(3). Then $c > \mathbb{Z}$ so $\mathbb{Z} + w'' + c \neq \mathbb{Z} + w''$. So by p.r.s.(1) we can find $d \in \widetilde{\Gamma}$ with $w'' \leq w'' + d \leq w'' + c$ with $\widetilde{\varrho}(w'' + d) = \widetilde{\varrho}(x'_{n'+1})$. Moreover, $\mathrm{st}\left(\frac{d}{y'_i}\right) = 0$ since $\mathrm{v}(d) \leq \mathrm{v}(c) < \mathrm{v}(y'_i)$ for all i. So setting $y'_{n'+1} = w'' + d$ we have that $\mathrm{st}\left(\frac{y'_{n'+1}}{y'_i}\right) = \mathrm{st}\left(\frac{w''+d}{y'_i}\right) = \mathrm{st}\left(\frac{w''}{y'_i}\right) + \mathrm{st}\left(\frac{d}{y'_i}\right) = \mathrm{st}\left(\frac{w''}{y'_i}\right) = \mathrm{st}\left(\frac{x'_{n'+1}}{y'_i}\right)$ for all i. We then have

$$\{x'_1, \ldots, x'_n, x'_{n'+1}\} \quad \text{and} \quad \{y'_1, \ldots, y'_n, y'_{n'+1}\}$$

strongly independent sets with $\widetilde{\varrho}(x'_{n'+1}) = \widetilde{\varrho}(y'_{n'+1})$. Moreover, $z \in \left\langle x'_1, \ldots, x'_{n'}, x'_{n'+1}\right\rangle$ so $z = q_1 x'_1 + \cdots + q_{n'} x'_{n'} + q_{n'+1} x'_{n'+1}$ for some $q_1, \ldots, q_{n'+1} \in \mathbb{Q}$ and we set $w = q_1 y'_1 + \cdots + q_{n'} y'_{n'} + q_{n'+1} y'_{n'+1}$. It is clear then that $\widetilde{\varrho}(z) = \widetilde{\varrho}(w)$ as required. In this second case we set $m = n' + 1$.

It remains to show, then, that $\mathrm{tp}(\bar{x}, z) = \mathrm{tp}(\bar{y}, w)$. The first thing to observe in order to show this is that

$$\mathrm{tp}(x'_1, \ldots, x'_m) = \mathrm{tp}(y'_1, \ldots, y'_m)\,. \tag{9.3}$$

This follows immediately from lemma 8.2.11 since by our construction we know that the sets

$$\{x'_1, \ldots, x'_m\} \quad \text{and} \quad \{y'_1, \ldots, y'_m\}$$

are strongly independent with $\widetilde{\varrho}(x'_i) = \widetilde{\varrho}(y'_i)$ and $\mathrm{st}\left(\frac{x'_i}{x'_j}\right) = \mathrm{st}\left(\frac{y'_i}{y'_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq m$.

We also know not only that

$$x_i = q_1 x'_1 + \cdots + q_m x'_m \text{ if and only if } y_i = q_1 y'_1 + \cdots + q_m y'_m \tag{9.4}$$

for all $q_1, \ldots, q_m \in \mathbb{Q}$ but also that

$$z = q_1 x'_1 + \cdots + q_m x'_m \text{ if and only if } w = q_1 y'_1 + \cdots + q_m y'_m \tag{9.5}$$

for all $q_1, \ldots, q_m \in \mathbb{Q}$ and that

$$\bar{x}, z \in \langle x'_1, \ldots, x'_m\rangle \quad \text{and} \quad \bar{y}, w \in \langle y'_1, \ldots, y'_m\rangle\,. \tag{9.6}$$

Now suppose $\gamma_0, \gamma_1, \dots, \gamma_{n+1} \in \mathbb{Z}$. Then

$$
\begin{aligned}
`\gamma_0 + \sum_{i=1}^{n} \gamma_i x_i + \gamma_{n+1} z > 0\text{'} \in \operatorname{tp}(\bar{x}, z) \\
&\iff \widetilde{\Gamma} \models \gamma_0 + \sum_{i=1}^{n} \gamma_i x_i + \gamma_{n+1} z > 0, \\
&\iff \widetilde{\Gamma} \models \gamma_0 + \sum_{i=1}^{m} q_i x'_i > 0 \text{ by 9.6}, \\
&\iff `\gamma_0 + \sum_{i=1}^{m} q_i x'_i > 0\text{'} \in \operatorname{tp}(x'_1, \dots, x'_m), \\
&\iff `\gamma_0 + \sum_{i=1}^{m} q_i y'_i > 0\text{'} \in \operatorname{tp}(y'_1, \dots, y'_m) \text{ by 9.3}, \\
&\iff \widetilde{\Gamma} \models \gamma_0 + \sum_{i=1}^{m} q_i y'_i + q_m y'_m > 0, \\
&\iff \widetilde{\Gamma} \models \gamma_0 + \sum_{i=1}^{n} \gamma_i y_i + \gamma_{n+1} w > 0, \text{ by 9.4 and 9.5}, \\
&\iff `\gamma_0 + \sum_{i=1}^{n} \gamma_i y_i + \gamma_{n+1} w > 0\text{'} \in \operatorname{tp}(\bar{y}, w).
\end{aligned}
$$

Moreover we also know that $\widetilde{\varrho}(x_i) = \widetilde{\varrho}(y_i)$ for $i \in \mathbb{N}$ with $1 \le i \le n$ and that $\widetilde{\varrho}(z) = \widetilde{\varrho}(w)$. It follows then that we must have $\operatorname{tp}(\bar{x}, z) = \operatorname{tp}(\bar{y}, w)$ as required. □

Corollary 9.3.10. If Γ is a countable, pseudo-recursively saturated model of Presburger arithmetic then Γ is homogeneous.

Proof. Again, use lemma 9.3.9 to produce a simple back-and-forth construction on $\widetilde{\Gamma}$ and then lift to an automorphism of Γ using proposition 3.2.3. □

Given theorem 9.3.4 this corollary shows there to be a strong connection between pseudo-recursive saturation and homogeneity.

Theorem 9.3.11. Suppose that Γ is a countable 2-homogeneous model of Presburger arithmetic. Then the following are equivalent:

1. G/G_{v} is transitive on V;

2. Γ is pseudo-recursively saturated.

Proof. (1) $\implies$ (2) Note first that for every value $v_1 \in \mathrm{V}$ there is some $\widetilde{g} \in G/G_{\mathrm{v}}$ and a larger value $v_2 > v_1$ such that v_2 is moved by g. For suppose otherwise. Then the values larger than v_1 would form a non-trivial proper block, contradicting the transitivity of V.

It then follows from the proof of theorem 9.3.4 [(3) $\implies$ (4)] that $\Gamma' = \Gamma$ (as taken from that proof), and hence that Γ is pseudo-recursively saturated.

(2) $\implies$ (1) By corollary 9.3.10 we know that Γ is homogeneous. So suppose that $v_1, v_2 \in \mathrm{V}$ are contained in some proper block. Since the block is proper we can find some v_3 outside of the block. By p.r.s.(1) we can find elements $\gamma_1, \gamma_2, \gamma_3$ with $\mathrm{v}(\gamma_i) = v_i$ for each $i = 1, 2, 3$ and $\varrho(\gamma_i) = \varrho(\gamma_j)$ for $i, j = 1, 2, 3$. Without loss of generality we suppose that $v_1 < v_2, v_3$. Then by homogeneity we can find some automorphism $g\colon \Gamma \to \Gamma$ which maps

$$g\colon \begin{array}{ccc} \gamma_1 & \mapsto & \gamma_1 \quad ; \\ \gamma_2 & \mapsto & \gamma_3 \quad . \end{array}$$

The existence of the map $\widetilde{g} \in g/G_{\mathrm{v}}$ clearly contradicts the fact that v_1 and v_2 are contained in a proper block. Hence G/G_{v} must be transitive on V. □

We have seen in proposition 9.3.6 that every recursively saturated model of Presburger arithmetic is pseudo-recursively saturated, but that the reverse is not necessarily the case. A reasonable question is that of what criterion can be added to pseudo-recursive saturation in order to necessitate recursive saturation? Are there any 'nice' conditions which will suffice? In the countable case we find that if $X, Y \subseteq \mathbb{N}$ are derived from $\mathrm{Res}(\Gamma)$ and $\mathrm{stQ}(\Gamma)$ respectively by some standard recursive encoding, then Γ is recursively saturated if and only if X and Y are Scott sets with $X = Y$ (a number of results from the next section are needed in order to show this; see Kaye [29, §13.1] for details of Scott sets). In general however, whilst from the discussion earlier there are clearly a number of necessary requirements which could be suggested, we do not as yet have a sufficient set.

In the next chapter we will use pseudo-recursive saturation for countable models to analyse automorphisms and their conjugates in detail.

Chapter 10

Automorphisms

10.1 Automorphisms of countable P.R.S. models

The following theorem provides a 'bare minimum' structure which can be used for constructing automorphisms. By adding extra criteria to the sets A and B more interesting automorphisms can be built; a technique which will be used extensively later on.

Theorem 10.1.1. Suppose Γ is a countable pseudo-recursively saturated model of Presburger arithmetic, and that we have strongly independent subsets of Γ

$$A = \{a_1, a_2, \dots, a_n\}, \qquad B = \{b_1, b_2, \dots, b_n\},$$

which satisfy the following:

1. $\varrho(a_i) = \varrho(b_i)$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$;
2. $\operatorname{st}\left(\frac{a_i}{a_j}\right) = \operatorname{st}\left(\frac{b_i}{b_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$.

Then there exists an automorphism $\theta\colon \Gamma \to \Gamma$ which maps a_i to b_i for all $i \in \mathbb{N}$ with $1 \leq i \leq n$.

Proof. The proof of this follows from lemma 8.2.11 and corollary 9.3.10. □

Although we are primarily concerned with automorphisms, it's worth noting that the previous theorem is really just a specific case of an application of the following theorem, which we state without proof.

Theorem 10.1.2. Let Γ_1, Γ_2 be Presburger groups with

1. $\text{Res}(\Gamma_1) \subseteq \text{Res}(\Gamma_2)$ and

2. $\text{stQ}(\Gamma_1) \subseteq \text{stQ}(\Gamma_2)$.

Suppose that Γ_2 is pseudo-recursively saturated, and that the sets $\{a_1, \ldots, a_n\} \subset \Gamma_1, \{b_1, \ldots, b_n\} \subset \Gamma_2$ are strongly independent.

Suppose further that

1. $\varrho(a_i) = \varrho(b_i)$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$,

2. $\text{st}\left(\frac{a_i}{a_j}\right) = \text{st}\left(\frac{b_i}{b_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$.

Then for all $a \in \Gamma_1 \setminus \langle a_1, \ldots, a_n \rangle$ there is some $a_{n+1} \in \Gamma_1$ and some $b_{n+1} \in \Gamma_2$ such that $a \in \langle a_1, \ldots, a_n, a_{n+1} \rangle$, with $\{a_1, \ldots, a_{n+1}\}$ and $\{b_1, \ldots, b_{n+1}\}$ strongly independent and so that the statements 1, 2 above still hold for these new sets.

Having established this theorem, it can be utilised by applying a simple back-and-forth construction in order to produce homomorphisms between Presburger groups Γ_1 and Γ_2. An immediate corollary of this is that models of Presburger arithmetic are completely characterized by their set of standard parts and residues: $\text{stQ}(\Gamma)$ and $\text{Res}(\Gamma)$ respectively.

The following result will be used extensively in the next two chapter on normal subgroups. However, despite its simplicity, it is surprisingly enlightening, and so it is useful to give it now in order to facilitate understanding of how the automorphisms act in general. It effectively states that automorphisms are required to 'scale' elements uniformly within valuation classes.

Lemma 10.1.3. Suppose $g \in G$ and $a, b \in \Gamma$ are such that $\text{st}\left(\frac{a}{b}\right) \in \mathbb{R} \setminus \{0\}$. Then

$$\text{st}\left(\frac{ag}{a}\right) = \text{st}\left(\frac{bg}{b}\right).$$

Proof. We have

$$\begin{aligned}\text{st}\left(\frac{ag}{a}\right) &= \text{st}\left(\frac{ag}{bg}\right) \cdot \text{st}\left(\frac{bg}{b}\right) \cdot \text{st}\left(\frac{b}{a}\right)\\ &= \text{st}\left(\frac{ag}{bg}\right) \cdot \text{st}\left(\frac{bg}{b}\right) \cdot \text{st}\left(\frac{bg}{ag}\right)\\ &= \text{st}\left(\frac{bg}{b}\right).\end{aligned}$$

All of the above are well defined since $\text{st}\left(\frac{a}{b}\right) \notin \{0, \pm\infty\}$ and since automorphisms preserve standard parts. □

In the next chapter we will consider the normal subgroups of the automorphism group $\mathrm{Aut}(\Gamma)$ where Γ is a countable pseudo-recursively saturated model of Presburger arithmetic. For brevity, this group will often be referred to simply as G. As we shall see in the next chapter, the most interesting automorphisms with regard to discovering normal subgroups would appear to be those which preserve values. The following theorem is therefore especially useful. It establishes the existence of an array of value preserving automorphisms, ensuring we have a varied supply of such automorphisms at our disposal.

Theorem 10.1.4. Let Γ be a countable pseudo-recursively saturated model of Presburger arithmetic and suppose we have an enumeration $\mathrm{v}(\gamma_1), \mathrm{v}(\gamma_2), \ldots$ of all the values of Γ with each element distinct. Suppose further that $h_1, h_2, \ldots$ are elements (not necessarily distinct) contained in $\mathrm{st}\,\mathbb{Q}\,\Gamma^{>0}$. Then there exists a value-preserving automorphism $g \colon \Gamma \to \Gamma$ such that for each $\gamma \in \mathrm{v}(\gamma_i)$,

$$\mathrm{st}\left(\frac{\gamma g}{\gamma}\right) = h_i.$$

What this theorem says is that we can find automorphisms by simply stating a scaling factor for each equivalence class of values. It should be noted, however, that the automorphism does not truly 'scale' the elements, but only maps to elements which are close to the actual scaled element if such were to exist (cf. definition 8.2.12). We can see that this is a necessary restriction, since unless an element has a residue of zero, any non-trivial multiple of it will have a different residue. Hence in general automorphisms cannot simple scale all elements.

Proof. Let $\gamma_1', \gamma_2', \ldots$ be an enumeration of $\widetilde{\Gamma} \setminus \{0\}$. We construct our required automorphism g using a back-and-forth argument on the elements of this enumeration.

At the m-th stage, suppose that for some $n \leq m$ we have

$$\{a_1, \ldots, a_n\} \quad \text{and} \quad \{b_1, \ldots, b_n\}$$

strongly independent sets with $\varrho(a_i) = \varrho(b_i)$ and $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$. Suppose further that for $i \in \mathbb{N}$ with $1 \leq i \leq n$, $\mathrm{st}\left(\frac{b_i}{a_i}\right) = h_j$ where $\mathrm{v}(a_i) = \mathrm{v}(\gamma_j)$ for some $j \in \mathbb{N}$.

We now consider the element γ_m' of our enumeration. If $\gamma_m' \in \langle a_1, \ldots, a_n \rangle$ we leave the enumerations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ as they are. The back-and-forth criteria just described will clearly still hold for these unaltered enumerations.

If on the other hand $\gamma'_m \notin \langle a_1, \ldots, a_n \rangle$ then we wish to find new elements in order to extend our enumerations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. By the Exchange Lemma (lemma 8.2.10) we can find some $a_{n+1} \in \widetilde{\Gamma}$ which is strongly independent of the elements $a_1, \ldots, a_n$ and such that $\gamma'_m \in \langle a_1, \ldots, a_n, a_{n+1} \rangle$.

Now either $\mathrm{v}(a_{n+1}) = \mathrm{v}(a_i)$ for some $i \in \mathbb{N}$ with $1 \leq i \leq n$ or there is no such i for which this holds. We consider the two cases separately.

If $\mathrm{v}(a_{n+1}) = \mathrm{v}(a_i)$ for some $i \in \mathbb{N}$ with $1 \leq i \leq n$ then

$$\mathrm{st}\left(\frac{a_{n+1}}{a_i}\right) = r \in \mathbb{R} \setminus \{0\}.$$

By p.r.s.(2) we can find $b'_{n+1} \in \widetilde{\Gamma}$ with

$$\mathrm{st}\left(\frac{b'_{n+1}}{b_i}\right) = r.$$

Now we choose $c \in \widetilde{\Gamma}$ such that $\mathrm{v}(0) < \mathrm{v}(c) < \mathrm{v}(b_i)$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$. We can do this by p.r.s.(3) since $b_i \neq 0$ for any i. Now $c > 0$ so $b'_{n+1} + c > b'_{n+1}$. Hence by p.r.s.(1) we can find d with $b'_{n+1} < b'_{n+1} + d < b'_{n+1} + c$ and such that $\varrho(b'_{n+1} + d) = \varrho(a_{n+1})$. We set $b_{n+1} = b'_{n+1} + d$. Now $\mathrm{v}(c) < \mathrm{v}(b_i)$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$, hence $\mathrm{v}(d) < \mathrm{v}(b_i)$ so that $\mathrm{st}\left(\frac{d}{b_i}\right) = 0$. In particular then,

$$\mathrm{st}\left(\frac{b_{n+1}}{b_i}\right) = \mathrm{st}\left(\frac{b'_{n+1} + d}{b_i}\right) = \mathrm{st}\left(\frac{b'_{n+1}}{b_i}\right) + \mathrm{st}\left(\frac{d}{b_i}\right) = \mathrm{st}\left(\frac{a_{n+1}}{a_i}\right).$$

We note that by our back-and-forth assumptions there is some $j \in \mathbb{N}$ for which

$$\mathrm{st}\left(\frac{b_i}{a_i}\right) = h_j$$

where $\mathrm{v}(a_i) = \mathrm{v}(\gamma_j)$. But

$$\mathrm{st}\left(\frac{b_{n+1}}{b_i}\right) = \mathrm{st}\left(\frac{a_{n+1}}{a_i}\right) = r$$

so

$$\mathrm{st}\left(\frac{b_{n+1}}{a_{n+1}}\right) = \mathrm{st}\left(\frac{b_{n+1}}{b_i}\right) \cdot \mathrm{st}\left(\frac{b_i}{a_i}\right) \cdot \mathrm{st}\left(\frac{a_i}{a_{n+1}}\right) = r h_j r^{-1} = h_j$$

and $\mathrm{v}(a_{n+1}) = \mathrm{v}(\gamma_j)$. Hence our new elements a_{n+1}, b_{n+1} satisfy the back-and-forth assumptions and we may extend our enumerations to

$$a_1, \ldots, a_n, a_{n+1} \quad \text{and} \quad b_1, \ldots, b_n, b_{n+1}$$

accordingly. Note that the element γ'_m which we originally considered is contained in the span $\langle a_1, \dots, a_n, a_{n+1}\rangle$.

We now consider the case where $\mathrm{v}(a_{n+1}) \neq \mathrm{v}(a_i)$ for any $i \in \mathbb{N}$ with $1 \leq i \leq n$. In this case one of the following must clearly hold:

1. $\mathrm{v}(a_{n+1}) < \mathrm{v}(a_k)$ for all $k \in \mathbb{N}$ with $1 \leq k \leq n$;
2. $\mathrm{v}(a_k) < \mathrm{v}(a_{n+1}) < \mathrm{v}(a_l)$ for some $k, l \in \mathbb{N}$ with $1 \leq k, l \leq n, k \neq l$;
3. $\mathrm{v}(a_k) < \mathrm{v}(a_{n+1})$ for all $k \in \mathbb{N}$ with $1 \leq k \leq n$.

Now for some (unique) $j \in \mathbb{N}$ we have $\mathrm{v}(a_{n+1}) = \mathrm{v}(\gamma_j)$. Since $h_j \in \mathrm{stQ}(\Gamma)$ we can certainly find some $b'_{n+1} \in \widetilde{\Gamma}$ such that

$$\mathrm{st}\left(\frac{b'_{n+1}}{a_{n+1}}\right) = h_j$$

by p.r.s.(2). We can also choose $c \in \widetilde{\Gamma}$ such that $\mathrm{v}(0) < \mathrm{v}(c) < \mathrm{v}(a_{n+1})$. We can do this by p.r.s.(3) since $a_{n+1} \neq 0$. Now $c > 0$ so $b'_{n+1} + c > b'_{n+1}$. Hence by p.r.s.(1) we can find d with $b'_{n+1} < b'_{n+1} + d < b'_{n+1} + c$ and such that $\varrho(b'_{n+1} + d) = \varrho(a_{n+1})$. We set $b_{n+1} = b'_{n+1} + d$. Now $\mathrm{v}(c) < \mathrm{v}(a_{n+1})$, hence $\mathrm{v}(d) < \mathrm{v}(a_{n+1})$ so that $\mathrm{st}\left(\frac{d}{a_{n+1}}\right) = 0$. In particular then,

$$\mathrm{st}\left(\frac{b_{n+1}}{a_{n+1}}\right) = \mathrm{st}\left(\frac{b'_{n+1} + d}{a_{n+1}}\right) = \mathrm{st}\left(\frac{b'_{n+1}}{a_{n+1}}\right) + \mathrm{st}\left(\frac{d}{b_{n+1}}\right) = \mathrm{st}\left(\frac{b'_{n+1}}{a_{n+1}}\right) = h_j.$$

Now clearly $\mathrm{v}(b_{n+1}) = \mathrm{v}(a_{n+1})$ and $\mathrm{v}(b_i) = \mathrm{v}(a_i)$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$. So it follows that one of

1. $\mathrm{v}(b_{n+1}) < \mathrm{v}(b_k)$ for all $k \in \mathbb{N}$ with $1 \leq k \leq n$;
2. $\mathrm{v}(b_k) < \mathrm{v}(b_{n+1}) < \mathrm{v}(a_l)$;
3. $\mathrm{v}(a_k) < \mathrm{v}(a_{n+1})$ for all $k \in \mathbb{N}$ with $1 \leq k \leq n$

must hold depending on which held for a_{n+1} as described in the previous list. It is clear from this that b_{n+1} is strongly independent of $b_1, \dots, b_n$ and we may also deduce that

$$\mathrm{st}\left(\frac{b_{n+1}}{b_i}\right) = \mathrm{st}\left(\frac{a_{n+1}}{a_i}\right)$$

for all $i \in \mathbb{N}$ with $1 \leq i \leq n$, as was required as part of the back-and-forth assumptions. In fact we have shown all of the back-and-forth assumptions to hold for the additional elements a_{n+1} and b_{n+1}, so we can extend our enumerations to

$$a_1, \dots, a_n, a_{n+1} \quad \text{and} \quad b_1, \dots, b_n, b_{n+1}$$

accordingly. As in the previous case we note that our original element γ'_m is contained in the span $\langle a_1, \dots, a_n, a_{n+1} \rangle$.

This completes the forth step. The back step is almost identical, although it becomes necessary to exchange the h_j for h_j^{-1}. It remains to show that we can construct an automorphism from our enumerations and that this automorphism satisfies our requirements. Let $\widetilde{g}$ be the map defined by the back-and-forth, so that

$$\widetilde{g} \colon a_i \mapsto b_i \text{ for all } i \in \mathbb{N}.$$

To show that $\widetilde{g}$ is well defined as an automorphism, consider any element $\gamma \in \widetilde{\Gamma} \setminus \{0\}$. Since the γ'_i form an enumeration of all the elements in $\widetilde{\Gamma} \setminus \{0\}$, we know that $\gamma = \gamma'_m$ for some $m \in \mathbb{N}$. By the manner in which the tuples were constructed we therefore know that

$$\gamma'_m \in \langle a_1, \dots, a_m \rangle$$

and hence that $\gamma'_m = q_1 a_1 + \cdots + q_m a_m$ for some $q_1, \cdots, q_m \in \mathbb{Q}$ not all zero. The mapping of γ'_m by $\widetilde{g}$ is therefore completely determined by the mapping of the a_i by $\widetilde{g}$:

$$\widetilde{g} \colon \gamma'_m \mapsto q_1 b_1 + \cdots + q_m b_m.$$

Moreover the sets $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_m\}$ are both strongly independent with consistent residues and standard parts, and hence by lemma 8.2.11 we know that

$$\mathrm{tp}(a_1, \dots, a_m) = \mathrm{tp}(b_1, \dots, b_m),$$

thus ensuring that $\mathrm{tp}(\gamma'_m) = \mathrm{tp}(\gamma'_m g)$. The map $\widetilde{g}$ will therefore be an automorphism as long as it is bijective.

For surjectivity, we note that by the back step, for every $\gamma = \gamma'_m \in \widetilde{\Gamma} \setminus \{0\}$ there exist $b_1, \dots, b_m$ in our enumeration such that $\gamma = q_1 b_1 + \cdots + q_m b_m$ for some $q_1, \dots, q_m \in \mathbb{Q}$ and hence there is some element which is mapped to it. To show that $\widetilde{g}$ is an injection, consider $\gamma'_{m_1}, \gamma'_{m_2} \in \widetilde{\Gamma}$ with $m_1 \neq m_2$. Then if $m = \max\{m_1, m_2\}$ we have $\gamma'_{m_1} = q_1 a_1 + \cdots + q_m a_m$ and $\gamma'_{m_2} = q'_1 a_1 + \cdots + q'_m a_m$ for some $q_1, \dots, q_m, q'_1, \dots, q'_m \in \mathbb{Q}$ and with $q_i \neq q'_i$ for at least one $i \in \mathbb{N}$ with $1 \leq i \leq m$. But then

$$\gamma'_{m_1} \widetilde{g} = q_1 b_1 + \cdots + q_m b_m \neq q'_1 b_1 + \cdots + q'_m b_m = \gamma'_{m_2} \widetilde{g}$$

since the $b_1, \dots, b_m$ are linearly independent. We see therefore that $\widetilde{g}$ is injective as required.

Clearly then, $\widetilde{g}$ is an automorphism. We claim that a lifting of this automorphism using proposition 3.2.3 to $g\colon \Gamma \to \Gamma$ will satisfy our requirements. For suppose $\gamma \in \widetilde{\Gamma}$. Then $\gamma = \gamma_i'$ for some $i \in \mathbb{N}$. We know by our back-and-forth construction that

$$\gamma \in \langle a_1, \dots, a_i \rangle ,$$

so $\gamma = q_1 a_1 + \cdots + q_i a_i$ for some $q_1, \dots, q_i \in \mathbb{Q} \setminus \{0\}$ and $\gamma\widetilde{g} = q_1 b_1 + \cdots + q_i b_i$.

Clearly $\varrho(\gamma) = \varrho(\gamma\widetilde{g})$ since $\varrho(a_j) = \varrho(b_j)$ for all $j \in \mathbb{N}$ with $1 \leq j \leq i$. Now $\mathrm{v}(\gamma) = \mathrm{v}(\gamma_k)$ for some $k \in \mathbb{N}$, so we must show that

$$\mathrm{st}\left(\frac{\gamma\widetilde{g}}{\gamma}\right) = \mathrm{st}\left(\frac{q_1 b_1 + \cdots + q_i b_i}{q_1 a_1 + \cdots + q_i a_i}\right) = h_k.$$

Now $a_1, \dots, a_i$ are strongly independent and $\mathrm{v}(a_j) = \mathrm{v}(b_j)$ for all $j \in \mathbb{N}$ with $1 \leq j \leq i$, so by lemma 8.2.5 we can let

$$\begin{aligned} v &= \mathrm{v}(q_1 a_1 + \cdots + q_i a_i) \\ &= \mathrm{v}(q_1 b_1 + \cdots + q_i b_i) \\ &= \max\{\, \mathrm{v}(q_j a_j) \colon 1 \leq j \leq i\} \\ &= \max\{\, \mathrm{v}(q_j b_j) \colon 1 \leq j \leq i\}. \end{aligned}$$

We separate the elements $b_1, \dots, b_i$ into those which have value v and those which do not. So suppose $b_{j_1}, b_{j_2}, \dots, b_{j_l}$ are precisely those which do have value v. Then

$$\mathrm{v}(b_{j_1}) = \mathrm{v}(b_{j_2}) = \cdots = \mathrm{v}(b_{j_l}) = v$$

and if $j \notin \{j_1, \dots, j_l\}$ then $\mathrm{v}(b_j) \neq v$. In fact $\mathrm{v}(b_j) \neq v$ implies that $\mathrm{v}(b_j) < v$, so we see that

$$\mathrm{st}\left(\frac{q_j b_j}{q_1 a_1 \cdots + q_i a_i}\right) = 0$$

for $j \notin \{j_1, \dots, j_l\}$. Moreover, because $\mathrm{v}(q_j b_j) \leq v = \mathrm{v}(\gamma)$ for all $j \in \mathbb{N}$ with $1 \leq j \leq i$

the following sums are well-defined:-

$$
\begin{aligned}
&\operatorname{st}\left(\frac{q_1 b_1 + \cdots + q_i b_i}{\gamma}\right) = \operatorname{st}\left(\frac{q_1 b_1}{\gamma}\right) + \cdots + \operatorname{st}\left(\frac{q_i b_i}{\gamma}\right) \\
&= \operatorname{st}\left(\frac{q_{j_1} b_{j_1}}{\gamma}\right) + \cdots + \operatorname{st}\left(\frac{q_{j_s} b_{j_s}}{\gamma}\right) + \cdots + \operatorname{st}\left(\frac{q_{j_l} b_{j_l}}{\gamma}\right) \\
&= \operatorname{st}\left(\frac{\gamma}{q_{j_1} b_{j_1}}\right)^{-1} + \cdots + \operatorname{st}\left(\frac{\gamma}{q_{j_s} b_{j_s}}\right)^{-1} + \cdots + \operatorname{st}\left(\frac{\gamma}{q_{j_l} b_{j_l}}\right)^{-1} \\
&= q_{j_1}\left(q_{j_1} \operatorname{st}\left(\frac{a_{j_1}}{b_{j_1}}\right) + \cdots + q_{j_l} \operatorname{st}\left(\frac{a_{j_1}}{b_{j_1}}\right) \operatorname{st}\left(\frac{a_{j_l}}{a_{j_1}}\right)\right)^{-1} + \cdots \\
&\quad \cdots + q_{j_s}\left(q_{j_1} \operatorname{st}\left(\frac{a_{j_s}}{b_{j_s}}\right) \operatorname{st}\left(\frac{a_{j_1}}{a_{j_s}}\right) + \cdots + q_{j_l} \operatorname{st}\left(\frac{a_{j_s}}{b_{j_s}}\right) \operatorname{st}\left(\frac{a_{j_l}}{a_{j_s}}\right)\right)^{-1} + \cdots \\
&\quad \cdots + q_{j_l}\left(q_{j_1} \operatorname{st}\left(\frac{a_{j_l}}{b_{j_l}}\right) \operatorname{st}\left(\frac{a_{j_1}}{a_{j_l}}\right) + \cdots + q_{j_l} \operatorname{st}\left(\frac{a_{j_l}}{b_{j_l}}\right)\right)^{-1} \\
&= q_{j_1}\left(q_{j_1} h_k^{-1} + \cdots + q_{j_l} h_k^{-1} \operatorname{st}\left(\frac{a_{j_l}}{a_{j_1}}\right)\right)^{-1} + \cdots \\
&\quad \cdots + q_{j_s}\left(q_{j_1} h_k^{-1} \operatorname{st}\left(\frac{a_{j_1}}{a_{j_s}}\right) + \cdots + q_{j_l} h_k^{-1} \operatorname{st}\left(\frac{a_{j_l}}{a_{j_1}}\right) \operatorname{st}\left(\frac{a_{j_1}}{a_{j_s}}\right)\right)^{-1} + \cdots \\
&\quad \cdots + q_{j_l}\left(q_{j_1} h_k^{-1} \operatorname{st}\left(\frac{a_{j_1}}{a_{j_l}}\right) + \cdots + q_{j_l} h_k^{-1} \operatorname{st}\left(\frac{a_{j_1}}{a_{j_l}}\right) \operatorname{st}\left(\frac{a_{j_l}}{a_{j_1}}\right)\right)^{-1} \\
&= q_{j_1} h_k \left(q_{j_1} + \cdots + q_{j_l} \operatorname{st}\left(\frac{a_{j_l}}{a_{j_1}}\right)\right)^{-1} + \cdots \\
&\quad \cdots + q_{j_s} h_k \operatorname{st}\left(\frac{a_{j_s}}{q_{j_1}}\right)\left(q_{j_1} + \cdots + q_{j_l} \operatorname{st}\left(\frac{a_{j_l}}{a_{j_1}}\right)\right)^{-1} + \cdots \\
&\quad \cdots + q_{j_l} h_k \operatorname{st}\left(\frac{a_{j_s}}{a_{j_1}}\right)\left(q_{j_1} + \cdots + q_{j_l} \operatorname{st}\left(\frac{a_{j_;}}{a_{j_1}}\right)\right)^{-1} \\
&= h_k \left(q_{j_1} + \cdots + q_{j_l} \operatorname{st}\left(\frac{a_{j_l}}{a_{j_1}}\right)\right)\left(q_{j_1} + \cdots + q_{j_l} \operatorname{st}\left(\frac{a_{j_l}}{a_{j_1}}\right)\right)^{-1} \\
&= h_k.
\end{aligned}
$$

Although these sums are somewhat tedious, it is nonetheless important to ensure that they are well defined and that we do not, for example, surreptitiously include the addition of $+\infty$ with $-\infty$. It is for this reason that they are given in full. When written out they do however clearly give us what is needed. The lifted map $g\colon \Gamma \to \Gamma$ will therefore satisfy our requirements as stated in the theorem. □

We shall discover in the next chapter that the following two automorphisms can be used to exhibit the existence of non-trivial, proper normal subgroups of G. Both automorphisms are given as corollaries to the previous theorem.

Corollary 10.1.5. Suppose Γ is a countable pseudo-recursively saturated model of Presburger arithmetic. Then there exists an automorphism $g_{\mathrm{st}}\colon \Gamma \to \Gamma$ such that $g \neq 1$ and with

$$\mathrm{st}\left(\frac{\gamma g_{\mathrm{st}}}{\gamma}\right) = 1$$

for every $\gamma \in \Gamma$.

Proof. Set $1 = h_1 = h_2 = h_3 = \cdots$. The previous theorem 10.1.4 says that we can find an automorphism $g\colon \Gamma \to \Gamma$ such that

$$\mathrm{st}\left(\frac{\gamma g}{\gamma}\right) = 1$$

for all $\gamma \in \Gamma$. However this alone will not suffice, since the construction in theorem 10.1.4 may simply generate the identity map, which we specifically wish to avoid. So begin by taking $a_1 = \gamma_1'$, the first element of the enumeration of $\widetilde{\Gamma} \setminus \{0\}$ in the proof of theorem 10.1.4. Clearly $\mathrm{v}(a_1) \neq 0$. So by p.r.s.(3) we can find $c \in \widetilde{\Gamma}$ with $0 \neq \mathrm{v}(c) < \mathrm{v}(a_1)$. Now $c > 0$ so $a_1 + c \neq a_1$ and so by p.r.s.(1) we can find some $0 < d \leq c$ for which $\widetilde{\varrho}(a_1 + d) = \widetilde{\varrho}(a_1)$. We set $b_1 = a_1 + d$ and clearly $b_1 \neq a_1$ as $d > 0$. Moreover since $\mathrm{v}(d) \leq \mathrm{v}(c) < \mathrm{v}(a_1)$ we have that $\mathrm{st}\left(\frac{d}{a_1}\right) = 0$, so in particular

$$\mathrm{st}\left(\frac{b_1}{a_1}\right) = \mathrm{st}\left(\frac{a_1 + d}{a_1}\right) = \mathrm{st}\left(\frac{a_1}{a_1}\right) + \mathrm{st}\left(\frac{d}{a_1}\right) = 1.$$

Now the elements a_1 and b_1 satisfy the requirements of the first step of the back-and-forth construction in the proof of theorem 10.1.4 with $1 = h_1 = h_2 = h_3 = \cdots$. We can therefore continue with the construction as it is set out in the proof of this theorem in order to produce an automorphism $g_{\mathrm{st}}\colon \Gamma \to \Gamma$ of the required sort. It is clear that $g_{\mathrm{st}} \neq 1$ since $a_1 g_{\mathrm{st}} = b_1$ and $a_1 \neq b_1$. □

Corollary 10.1.6. Suppose Γ is a countable pseudo-recursively saturated model of Presburger arithmetic with $\mathrm{stQ}(\Gamma)^* \neq \{1\}$. Then there exists an automorphism $g_{\mathrm{v}}\colon \Gamma \to \Gamma$ such that $\mathrm{v}(\gamma) = \mathrm{v}(\gamma g_{\mathrm{v}})$ for all $\gamma \in \Gamma$ but

$$\mathrm{st}\left(\frac{\gamma' g_{\mathrm{v}}}{\gamma'}\right) \neq 1$$

for at least one $\gamma' \in \Gamma$.

Proof. Since $\mathrm{stQ}(\Gamma)^* \neq \{1\}$ we can find some $h \in \mathrm{stQ}(\Gamma)^*$ with $h \neq 1$. Simply set $h = h_1 = h_2 = h_3 = \cdots$ and apply theorem 10.1.4 to find an automorphism $g_{\mathrm{v}}\colon \Gamma \to \Gamma$ of the required sort. □

10.2 Conjugates of automorphisms which defy values

As our aim is to examine the closed normal subgroups of G, we will look closely at the effect of conjugation of automorphisms. We will do this in two parts: first in this section for the value-defying automorphisms (those which map at least one element to an element with a different value) and subsequently in section 10.4 for value-preserving automorphisms (those which map elements only to elements with the same value).

The standard parts are pivotal in this analysis, so it will benefit us to define a term covering the set of standard parts mapped as ratios by an automorphism. We use the same notation as that for the set of standard parts existing in a model (cf. notation on page 54) as the distinction should always be clear from the context.

Definition 10.2.1. Let H be a subset of G. Then we define

$$\mathrm{stQ}(H) = \left\{ r \in \mathbb{R} : \exists\, \gamma \in \Gamma, h \in H \,\mathrm{st}\left(\frac{\gamma h}{\gamma}\right) = r \right\}.$$

If $H = \{h\}$ we may write $\mathrm{stQ}(h)$ instead of $\mathrm{stQ}(\{h\})$.

This can be extended to apply to the factored Presburger group $\widetilde{\Gamma}$ in the obvious manner.

Lemma 10.2.2. Suppose H is a subset of G_{v}. Then $\mathrm{stQ}(\langle H^G \rangle) = \langle \mathrm{stQ}(H) \rangle$ in $\mathbb{R}^*$.

Proof. We begin by showing that $\mathrm{stQ}(\langle H^G \rangle) \subseteq \langle \mathrm{stQ}(H) \rangle$. So suppose that $r = \mathrm{st}\left(\frac{\gamma h}{\gamma}\right)$ for some $\gamma \in \Gamma$ and $h \in \langle H^G \rangle$. Hence $h = h_1^{g_1} h_2^{-g_2} \ldots h_{2n-1}^{g_{2n-1}} h_{2n}^{-g_{2n}}$ where $h_1, \ldots, h_{2n} \in H$ (all of which preserve values) and $g_1, \ldots, g_{2n} \in G$. We will use induction to show that in this case $r \in \langle \mathrm{stQ}(H) \rangle$.

For the base case, taking $n = 1$, we have that

$$\begin{aligned}
\mathrm{st}\left(\frac{\gamma_1 h_1^{g_1} h_2^{-g_2}}{\gamma_1}\right) &= \mathrm{st}\left(\frac{\gamma_1 g_1 h_1 g_1^{-1}}{\gamma_1}\right) \cdot \mathrm{st}\left(\frac{\gamma_1 g_1 h_1 g_1^{-1} g_2 h_2^{-1} g_2^{-1}}{\gamma_1 g_1 h_1 g_1^{-1}}\right) \\
&= \mathrm{st}\left(\frac{\gamma_1 g_1 h_1}{\gamma_1 g_1}\right) \cdot \mathrm{st}\left(\frac{\gamma_2 g_2 h_2}{\gamma_2 g_2}\right)^{-1} \\
&= r_1 r_2^{-1}
\end{aligned}$$

for $r_1, r_2 \in \mathrm{stQ}(H)$. Hence $r_1 r_2^{-1} \in \langle \mathrm{stQ}(H) \rangle$ as required.

Now suppose the statement is true for some $n \in \mathbb{N}$, *i.e.* that

$$\mathrm{st}\left(\frac{\gamma h_1^{g_1} h_2^{-g_1} \dots h_{2n-1}^{g_{2n-1}} h_{2n}^{-g_{2n}}}{\gamma}\right) = r \in \langle \mathrm{stQ}(H) \rangle$$

where $h_1, \dots, h_{2n} \in H$ and $g_1, \dots, g_{2n} \in G$. Now choose further automorphisms $h_{2n+1}, h_{2n+2} \in H$ and $g_{2n+1}, g_{2n+2} \in G$. We then have that

$$\begin{aligned}
&\mathrm{st}\left(\frac{\gamma h_1^{g_1} h_2^{-g_2} \dots h_{2n-1}^{g_{2n-1}} h_{2n}^{-g_{2n}} h_{2n+1}^{g_{2n+1}} h_{2n+2}^{-g_{2n+2}}}{\gamma}\right) \\
&\quad = \mathrm{st}\left(\frac{\gamma h_1^{g_1} h_2^{-g_2} \dots h_{2n-1}^{g_{2n-1}} h_{2n}^{-g_{2n}}}{\gamma}\right) \cdot \mathrm{st}\left(\frac{\gamma h_1^{g_1} h_2^{-g_2} \dots h_{2n-1}^{g_{2n-1}} h_{2n}^{-g_{2n}} h_{2n+1}^{g_{2n+1}} h_{2n+2}^{-g_{2n+2}}}{\gamma h_1^{g_1} h_2^{-g_2} \dots h_{2n-1}^{g_{2n-1}} h_{2n}^{-g_{2n}}}\right) \\
&\quad = r \cdot \mathrm{st}\left(\frac{\gamma h_{2n+1}^{g_{2n+1}}}{\gamma}\right) \cdot \mathrm{st}\left(\frac{\gamma h_{2n+1}^{g_{2n+1}} h_{2n+2}^{-g_{2n+2}}}{\gamma h_{2n+1}^{g_{2n+1}}}\right) \\
&\quad = r \cdot \mathrm{st}\left(\frac{\gamma g_{2n+1} h_{2n+1}}{\gamma g_{2n+1}}\right) \cdot \mathrm{st}\left(\frac{\gamma h_{2n+1}^{g_{2n+1}} g_{2n+2} h_{2n+2}}{\gamma h_{2n+1}^{g_{2n+1}} g_{2n+2}}\right)^{-1} \\
&\quad \in \langle \mathrm{stQ}(H) \rangle \,.
\end{aligned}$$

The inductive step therefore holds and we are done.

It remains to show that $\langle \mathrm{stQ}(H) \rangle \subseteq \mathrm{stQ}(\langle H^G \rangle)$. So take $r \in \langle \mathrm{stQ}(H) \rangle$. Then $r = r_1 r_2^{-1} \dots r_{2n-1} r_{2n}^{-1}$ where $r_1, \dots, r_{2n} \in \mathrm{stQ}(H)$. In other words, $r_i = \mathrm{st}\left(\frac{\gamma_i h_i}{\gamma_i}\right)$ where $\gamma_i \in \Gamma$ and $h_i \in H$ for $i \in \mathbb{N}$ with $1 \leq i \leq 2n$. We claim that $r \in \mathrm{stQ}(\langle H^G \rangle)$ and again we show this by induction.

So for the base case let $n = 1$. Then

$$r = \mathrm{st}\left(\frac{\gamma_1 h_1}{\gamma_1}\right) \cdot \mathrm{st}\left(\frac{\gamma_2 h_2^{-1}}{\gamma_2}\right).$$

By p.r.s.(2) and lemma 10.1.3 we may assume that $\varrho(\gamma_1) = \varrho(\gamma_2)$, hence by theorem 10.1.1 we can find an automorphism $g \in G$ such that $g \colon \gamma_1 \mapsto \gamma_2$. But then

$$\begin{aligned}
r &= \mathrm{st}\left(\frac{\gamma_1 h_1}{\gamma_1}\right) \cdot \mathrm{st}\left(\frac{\gamma_1 g h_2^{-1}}{\gamma_1 g}\right) \\
&= \mathrm{st}\left(\frac{\gamma_1 h_1}{\gamma_1}\right) \cdot \mathrm{st}\left(\frac{\gamma_1 g h_2^{-1} g^{-1}}{\gamma_1}\right) \\
&= \mathrm{st}\left(\frac{\gamma_1 h_1 g h_2^{-1} g^{-1}}{\gamma_1}\right) \\
&\in \mathrm{stQ}(\langle H^G \rangle).
\end{aligned}$$

Now suppose the statement is true for some $n \in \mathbb{N}$, *i.e.* that

$$r_1 r_2^{-1} \dots r_{2n-1} r_{2n}^{-1} = \mathrm{st}\left(\frac{\gamma h}{\gamma}\right) \in \mathrm{stQ}(\langle H^G \rangle)$$

for some $r_1, \dots, r_{2n} \in \mathrm{stQ}(H)$. Taking $r_{2n+1}, r_{2n+2} \in \mathrm{stQ}(H)$ we then have that

$$r_1 r_2^{-1} \dots r_{2n-1} r_{2n}^{-1} r_{2n+1} r_{2n+2}^{-1} = \mathrm{st}\left(\frac{\gamma h}{\gamma}\right) \cdot \mathrm{st}\left(\frac{\gamma_{2n+1} h_{2n+1}}{\gamma_{2n+1}}\right) \cdot \mathrm{st}\left(\frac{\gamma_{2n+2} h_{2n+2}^{-1}}{\gamma_{2n+2}}\right).$$

But again by p.r.s.(2) and lemma 10.1.3 we may assume that $\varrho(\gamma) = \varrho(\gamma_{2n+1}) = \varrho(\gamma_{2n+2})$, hence by theorem 10.1.1 we can find automorphisms $g_{2n+1}, g_{2n+2} \in G$ such that $g_{2n+1} \colon \gamma \mapsto \gamma_{2n+1}$ and $g_{2n+2} \colon \gamma \mapsto \gamma_{2n+2}$. Then

$$\begin{aligned} r_1 r_2^{-1} \dots r_{2n-1} r_{2n}^{-1} r_{2n+1} r_{2n+2}^{-1} &= \mathrm{st}\left(\frac{\gamma h}{\gamma}\right) \cdot \mathrm{st}\left(\frac{\gamma g_{2n+1} h_{2n+1}}{\gamma g_{2n+1}}\right) \cdot \mathrm{st}\left(\frac{\gamma g_{2n+2} h_{2n+2}^{-1}}{\gamma g_{2n+2}}\right) \\ &= \mathrm{st}\left(\frac{\gamma h g_{2n+1} h_{2n+1} g_{2n+1}^{-1} g_{2n+2} h_{2n+2}^{-1} g_{2n+2}^{-1}}{\gamma}\right) \\ &\in \mathrm{stQ}(\langle H^G \rangle). \end{aligned}$$

The inductive step therefore holds and we are done. □

The result we need for value-defying automorphism now follows. It effectively tells us that using a conjugate of a value-defying automorphism we can map any tuple of elements to any other sensible tuple of elements. It is a consequence of this that any closed normal subgroup containing a value-defying automorphism must necessarily be the whole of G.

Theorem 10.2.3. Let Γ be a pseudo-recursively saturated model of Presburger arithmetic and suppose $a_1, \dots, a_n \in (\Gamma \setminus \mathbb{Z})_{>0}$ and $b_1, \dots, b_n \in (\Gamma \setminus \mathbb{Z})_{>0}$ are such that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. Suppose further that $h \in G$ does not preserve values. Then there exist $g_1, g_2 \in G$ such that

$$\bar{a} h^{\pm g_1} h^{\mp g_2} = \bar{b}.$$

Proof. We may suppose without loss of generality that $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$ and by theorem 9.3.8 we may also assume that the sets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are each strongly independent. Since h does not preserve values there is some $x \in \Gamma_{>0}$ with either $\mathrm{st}\left(\frac{xh}{x}\right) = \infty$ or $\mathrm{st}\left(\frac{xh^{-1}}{x}\right) = \infty$. We will suppose the former is the case and for the latter simply swap h for h^{-1} in the following arguments. We therefore have that $\mathrm{v}(x) < \mathrm{v}(xh)$. Let $y = xh$. Then for any $\gamma \in \Gamma_{>0}$ with $\mathrm{v}(x) < \mathrm{v}(\gamma) < \mathrm{v}(y)$ it is clear that $\gamma h > xh = y$ and so $\mathrm{st}\left(\frac{\gamma h}{\gamma}\right) = \infty$. By density of values (p.r.s.(3)) there are clearly infinitely many such γ with distinct values.

Now let $x_1, \dots, x_n$ be such that

$$x < x_1 < x_2 < \cdots < x_{n-1} < x_n < y.$$

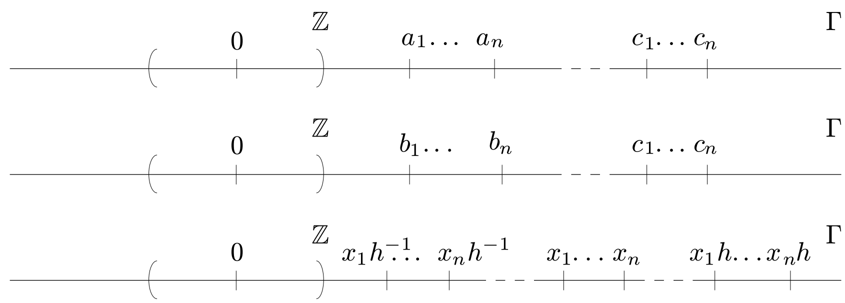

Figure 10.1: Ordering of strongly independent elements.

By p.r.s.(1) and p.r.s.(2) we can choose such $x_1, \dots, x_n$ with the correct ordering, residues and standard parts so that $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{a})$. Moreover, by p.r.s.(3) we can also choose $c_1 < \dots < c_n$ with $\mathrm{tp}(\bar{c}) = \mathrm{tp}(\bar{a})$ so that $\mathrm{v}(c_i) > \mathrm{v}(a_j)$ and $\mathrm{v}(c_i) > \mathrm{v}(b_j)$ for any $i, j \in \mathbb{N}$ with $1 \le i, j \le n$. We will then have

$$\mathrm{st}\left(\frac{x_i}{x_j}\right) = \mathrm{st}\left(\frac{x_i h}{x_j h}\right) = \mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right) = \mathrm{st}\left(\frac{c_i}{c_j}\right),$$

and

$$\varrho(x_i) = \varrho(x_i h) = \varrho(a_i) = \varrho(b_i) = \varrho(c_i),$$

for all $i, j \in \mathbb{N}$ with $1 \le i < j \le n$. In particular, it is the case that

$$\{x_1, \dots, x_n, x_1 h, \dots, x_n h\} \quad \text{and} \quad \{a_1, \dots, a_n, c_1, \dots, c_n\}$$

are each strongly independent sets. Similar arguments ensure that

$$\{x_1 h, \dots, x_n h, x_1, \dots, x_n\} \quad \text{and} \quad \{c_1, \dots, c_n, b_1, \dots, b_n\}$$

are also each strongly independent sets. Figure 10.1 gives a pictorial representation of the ordering of these strongly independent sets on three copies of Γ. The dashed line indicates a 'valuation gap', *i.e.* where a difference of value is guaranteed between the points either side of the dashed interval.

By theorem 10.1.1 we can therefore create automorphisms $g_1 \colon \Gamma \to \Gamma$ and $g_2 \colon \Gamma \to \Gamma$ which map elements as follows:

$$g_1\colon \begin{array}{rcll} x_1 & \mapsto & a_1 & ; \\ \vdots & & \vdots & \\ x_n & \mapsto & a_n & ; \\ x_1h & \mapsto & c_1 & ; \\ \vdots & & \vdots & \\ x_nh & \mapsto & c_n & \end{array} \qquad \text{and} \qquad g_2\colon \begin{array}{rcll} x_1h & \mapsto & c_1 & ; \\ \vdots & & \vdots & \\ x_nh & \mapsto & c_n & ; \\ x_1 & \mapsto & b_1 & ; \\ \vdots & & \vdots & \\ x_n & \mapsto & b_n & . \end{array}$$

We then have $\bar{a}h^{g_1}h^{-g_2} = \bar{c}h^{-g_2} = \bar{b}$ as required. □

10.3 Coloured sets of ordered values

Before investigating value-preserving automorphisms, it will be useful to consider an analogy. We will discover presently that when considering closed normal subgroups of G we are particularly interested in tuples $(r_1, \dots, r_n) \in (\mathrm{stQ}(\Gamma)_{>0})^n$ where

$$(r_1, \dots, r_n) = \left(\mathrm{st}\left(\frac{\gamma_1 g}{\gamma_1}\right), \dots, \mathrm{st}\left(\frac{\gamma_n g}{\gamma_n}\right)\right)$$

for some $\gamma_1, \dots, \gamma_n \in \Gamma$ and $g \in G$. However, such tuples can also be seen as projections of the coloured set of ordered values of an automorphism g. Since the analogy between the two may turn out to be conceptually beneficial, we will describe exactly what is meant by this.

Definition 10.3.1. Recall from definition 8.2.2 that $\mathrm{V} = \widetilde{\Gamma}/\!\equiv$ is the set of values of $\widetilde{\Gamma}$. By p.r.s.(3) we know that V is a dense linear order with least point 0 and no greatest point. If $g \in G_{\mathrm{v}}$ we assign a colouring $c_g\colon \mathrm{V} \to \mathbb{R}$ to the elements of V so that if $\gamma \in \widetilde{\Gamma}$ is such that $\mathrm{v}(\gamma) = v$, then

$$c_g\colon v \mapsto \mathrm{st}\left(\frac{\gamma g}{\gamma}\right).$$

The fact that this is well defined follows from lemma 10.1.3. We call V with its associated colouring the **coloured set of ordered values determined by** g.

We can give this an even more explicit meaning using the following:

Definition 10.3.2. Define $\mathrm{stQ}^{\mathrm{V}}$ to be the product

$$\mathrm{stQ}^{\mathrm{V}} = \prod_{v \in \mathrm{V}} \mathrm{stQ}(\Gamma)_v,$$

where each $\mathrm{stQ}(\Gamma)_v$ is a copy of $\mathrm{stQ}(\Gamma)_{>0}$ indexed by an element v of V. Thus $\mathrm{stQ}^{\mathrm{V}}$ is equivalent to the set of all functions from V to $\mathrm{stQ}(\Gamma)_{>0}$.

We can define the function $\theta\colon G_{\mathrm{v}} \to \mathrm{stQ}^{\mathrm{V}}$ to be the homomorphism of semigroups given by

$$\theta\colon g \mapsto \left(\dots, \mathrm{st}\left(\frac{\gamma_v g}{\gamma_v}\right), \dots\right) \in \mathrm{stQ}^{\mathrm{V}},$$

where $\gamma_v \in \Gamma$ is any instance from the equivalence class $v \in \mathrm{V}$, *i.e.* with $\mathrm{v}(\gamma_v) = v$. A neater, but perhaps slightly more obscure way to express this would be to say that

$$\theta\colon g \mapsto c_g$$

where c_g is the colouring as described above in definition 10.3.1. In order to show that we have a homomorphism we consider elements $\theta(g_1)$ and $\theta(g_2)$ of $\mathrm{stQ}^{\mathrm{V}}$ for some $g_1, g_2 \in G_{\mathrm{v}}$. Then we see that

$$\begin{aligned}
\theta(g_1 g_2) &= \prod_{v\in \mathrm{V}} \mathrm{st}\left(\frac{\gamma_v g_1 g_2}{\gamma_v}\right) = \prod_{v\in \mathrm{V}} \mathrm{st}\left(\frac{\gamma_v g_1}{\gamma_v}\right) \cdot \mathrm{st}\left(\frac{\gamma_v g_2}{\gamma_v}\right) \\
&= \prod_{v\in \mathrm{V}} \mathrm{st}\left(\frac{\gamma_v g_1}{\gamma_v}\right) \prod_{v\in \mathrm{V}} \mathrm{st}\left(\frac{\gamma_v g_2}{\gamma_v}\right) \\
&= \theta(g_1) \cdot \theta(g_2)
\end{aligned}$$

so that multiplication occurs pointwise.

Now the analogy between colourings and finite tuples of standard parts arises by considering projections of $\mathrm{stQ}^{\mathrm{V}}$ of the form

$$\pi_{V_1}\colon \mathrm{stQ}^{\mathrm{V}} \to \prod_{v\in V_1} (\mathrm{stQ}(\Gamma)_{>0})_v$$

where $V_1 = \{v_1, \dots, v_n\}$ is some finite subset of V. This is not quite the whole story, since as a general rule for a tuple

$$(r_1, \dots, r_n) = \left(\mathrm{st}\left(\frac{\gamma_1 g}{\gamma_1}\right), \dots, \mathrm{st}\left(\frac{\gamma_n g}{\gamma_n}\right)\right)$$

we may not be choosing $\mathrm{v}(\gamma_1) < \dots < \mathrm{v}(\gamma_n)$, whilst for the projection this will always be the case. We will discover towards the end of this thesis—in section 12.4—that the distinction for closed normal subgroups of G is of little consequence, however for the time being it is something which must be taken into account.

We can also define the action of g_2 on $\theta(g_1)$ simply as $\theta(g_1)g_2 = \theta(g_1 g_2)$.

The above applies only for $g_2 \in G_{\mathrm{v}}$ which preserve values. For $g_2 \in G \setminus G_{\mathrm{v}}$ we can also consider the effect of applying g_2 to some element $\theta(g_1) \in \mathrm{stQ}^{\mathrm{V}}$. In this case the result is similar to applying an automorphism of the underlying dense linear order of V, although this only holds fully if $\mathrm{stQ}(g_2) \subseteq \{0, \pm\infty\}$, which will not generally be the case for elements of $G \setminus G_{\mathrm{v}}$. We will not define this explicitly, but simply point out that the effect will be relevant in later constructions.

The description given in this section will hopefully serve to clarify some of the constructions which we are now going on to prove, and equivalent statements of results will be given in order to facilitate this.

10.4 Conjugates of automorphisms which preserve values

We will now turn our attention to value-preserving automorphisms. Because we are interested in normal subgroups, it turns out that we can consider the value-preserving automorphisms in complete isolation from the value-defying ones. This fact follows from the next lemma:

Lemma 10.4.1. Suppose $h \in \widetilde{G}_{\mathrm{v}}$, and $g \in \widetilde{G}$ is arbitrary. Then $\mathrm{v}(\gamma g^{-1} h g) = \mathrm{v}(\gamma)$ for all $\gamma \in \widetilde{\Gamma}$.

Proof. The proof is straightforward as we simply need to notice that $\mathrm{v}(\gamma g^{-1} h) = \mathrm{v}(\gamma g^{-1})$. □

Here we see that, given a value-preserving automorphism, any conjugate of it will also preserve values. In particular, if we take the normal subgroup generated by a set of value-preserving automorphisms, we know immediately that it will be a subgroup of G_{v}, in contrast to a set containing a value-defying automorphism, for which the closed normal subgroups generated by it will always be the whole of G as shown in section 10.2. A much more comprehensive version of lemma 10.4.1 can be found as proposition 11.3.1, in which it is shown that G_{v} is a closed normal subroup of G.

So we will continue by considering conjugates of value-preserving automorphisms in more detail. The next result indicates a type of automorphism we are able to construct whilst taking into account the coloured set of ordered values of the model. It and the lemma following it will be used in order to show the flexibility we have when applying conjugation to value-preserving automorphisms.

Proposition 10.4.2. Suppose $a_1, \dots, a_n$ and $b_1, \dots, b_n \in \widetilde{\Gamma}$ are positive elements with $0 < \mathrm{v}(a_n) < \mathrm{v}(a_{n-1}) < \cdots < \mathrm{v}(a_1)$ and $0 < \mathrm{v}(b_n) < \mathrm{v}(b_{n-1}) < \cdots < \mathrm{v}(b_1)$, and that $\widetilde{\varrho}(a_i) = \widetilde{\varrho}(b_i)$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$. Let

$$W_{\bar{a}} = \{\, \mathrm{v}(a) : 0 < \mathrm{v}(a) < \mathrm{v}(a_n)\}$$
$$W_{\bar{b}} = \{\, \mathrm{v}(b) : 0 < \mathrm{v}(b) < \mathrm{v}(b_n)\}$$

be sets of values considered as ordered sets with an isomorphism $f\colon W_{\bar{a}} \to W_{\bar{b}}$. Then there is $g \in \widetilde{G}$ with $g\colon \bar{a} \mapsto \bar{b}$ and such that $\mathrm{v}(xg) = f(\mathrm{v}(x))$ for all $x \in \mathrm{v}^{-1}(W_{\bar{a}})$.

Moreover if

$$\alpha\colon W_{\bar{a}} \to \mathrm{v}^{-1}(W_{\bar{a}})$$
$$\beta\colon W_{\bar{b}} \to \mathrm{v}^{-1}(W_{\bar{b}})$$

are such that $\alpha \circ \mathrm{v} = \mathrm{id}_{W_{\bar{a}}}$ and $\beta \circ \mathrm{v} = \mathrm{id}_{W_{\bar{b}}}$ then g can be chosen such that

$$g(\alpha(x)) \frown \beta(f(x))$$

for all $x \in W_{\bar{a}}$.

This proposition is particularly useful when considering f as an isomorphism preserving colouring when $W_{\bar{a}}$ and $W_{\bar{b}}$ are considered as coloured sets of ordered values. In this case the proposition effectively tells us that if $v_n < \cdots < v_1 \in V$ and $v'_n < \cdots < v'_1 \in V$ and if we set $V_1 = \{\, v \in V : v < v_n\}$ and $V_2 = \{\, v \in V : v < v'_n\}$ then there is an automorphism $g \in G$ such that the effect of g on V (in other words, the map g/V) is to send v_i to v'_i for all $i \in \mathbb{N}$ with $1 \leq i \leq n$ and so that $g/\mathrm{V}\colon V_1 \to V_2$ is an isomorphism of the coloured set of ordered values.

Proof. Before obtaining the map g using a back-and-forth construction we will need various enumerations. So let $(v_i)_{i\in\mathbb{N}}$ be an enumeration of $W_{\bar{a}}$ and $(v'_i)_{i\in\mathbb{N}}$ an enumeration of $W_{\bar{b}}$. Now define $\gamma_i = \alpha(v_i)$ and $\gamma'_i = \beta(v'_i)$. It is clear that we wish to construct g so that $\gamma_i g \frown \beta(f(\mathrm{v}(\gamma_i)))$ for all $i \in \mathbb{N}$, and that $\beta(f(\mathrm{v}(\gamma_i))) = \gamma'_j$ for some $j \in \mathbb{N}$. We use these facts to produce the back-and-forth construction.

So we wish to find strongly independent sets

$$\{\delta_1, \dots, \delta_n\} \quad \text{and} \quad \{\delta'_1, \dots, \delta'_n\}$$

such that

1. $\varrho(\delta_i) = \varrho(\delta_i')$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$;

2. $\operatorname{st}\left(\frac{\delta_i}{\delta_j}\right) = \operatorname{st}\left(\frac{\delta_i'}{\delta_j'}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$;

3. if $\mathrm{v}(\delta_i) = \mathrm{v}(\gamma_j)$ for some $i, j \in \mathbb{N}$ with $1 \leq i \leq n$ then $\delta_k \frown \gamma_j$ for some $k \leq i$ and $\delta_k' \frown \beta(f(\mathrm{v}(\gamma_j)))$.

So let $e_1, e_2, \ldots$ be an enumeration of the elements of $\widetilde{\Gamma}$. To begin we set $\delta_i = a_i$ and $\delta_i' = b_i$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$. It is clear that this satisfies the back-and-forth criteria. Now at the $(m-n-1)$-th stage, suppose we have

$$\{\delta_1, \ldots, \delta_{m-1}\} \quad \text{and} \quad \{\delta_1', \ldots, \delta_{m-1}'\}$$

which satisfy the criteria. We consider the smallest i such that $e_i \notin \langle \delta_1, \ldots, \delta_{m-1} \rangle$. Now either e_i is strongly independent of $\{\delta_1, \ldots, \delta_{m-1}\}$, in which case set $e = e_i$, or it is not. If not, then by the Exchange Lemma (lemma 8.2.10) we can find some e which is strongly independent and such that $e_i \in \langle \delta_1, \ldots, \delta_{m-1}, e \rangle$. We now consider the element e in more detail, there being a number of different cases to consider.

It may be that $\mathrm{v}(e) = \mathrm{v}(\delta_j)$ for some $j \in \mathbb{N}$ with $1 \leq j \leq m-1$. In this case, we know that $\operatorname{st}\left(\frac{e}{\delta_j}\right) = r$ for some $r \in \mathbb{R} \setminus \{0\}$. So we let $\delta_m = e$ and set δ_m' to be some element of $\widetilde{\Gamma}$ such that $\widetilde{\varrho}(\delta_m') = \widetilde{\varrho}(\delta_m)$ and with $\operatorname{st}\left(\frac{\delta_m'}{\delta_j'}\right) = r$. We know that such an element exists by p.r.s.(1) and p.r.s.(2). It is clear that this extension to $\{\delta_1, \ldots, \delta_m\}$ and $\{\delta_1', \ldots, \delta_m'\}$ satisfies the criteria and completes the back-and-forth step.

Alternatively it may be that there is no $j \in \mathbb{N}$ with $1 \leq j \leq m-1$ for which $\mathrm{v}(e) = \mathrm{v}(\delta_j)$. In this case we consider whether $\mathrm{v}(e) = \mathrm{v}(\gamma_j)$ for some $j \in \mathbb{N}$. If not then we know that $\mathrm{v}(e) > \mathrm{v}(a_n)$. Thus we may let $\delta_m = e$ and δ_m' be any element with $\widetilde{\varrho}(\delta_m') = \widetilde{\varrho}(\delta_m)$ and having consistent ordering with the values of the elements $\delta_1, \ldots, \delta_{m-1}$. For example, if $\mathrm{v}(\delta_s) < \mathrm{v}(\delta_m) < \mathrm{v}(\delta_t)$ we must ensure that $\mathrm{v}(\delta_s') < \mathrm{v}(\delta_m') < \mathrm{v}(\delta_t')$. We know we can do this by denseness of values (p.r.s.(3)). In this case the back-and-forth step is complete.

So suppose that $\mathrm{v}(e) = \mathrm{v}(\gamma_j)$ for some $j \in \mathbb{N}$. Again, we have two cases to consider. Either $\{\gamma_j, e\}$ is strongly independent or not. If it is then we may set $\delta_m = \gamma_j$ and $\delta_{m+1} = e$. Now let $k \in \mathbb{N}$ be the value for which $\gamma_k' = \beta(f(\mathrm{v}(\gamma_j)))$. Then set $\delta_m' \frown \gamma_k'$ with $\widetilde{\varrho}(\delta_m') = \widetilde{\varrho}(\delta_m)$ and δ_{m+1}' to be any element for which $\widetilde{\varrho}(\delta_{m+1}') = \widetilde{\varrho}(\delta_{m+1})$ and so that $\operatorname{st}\left(\frac{\delta_{m+1}}{\gamma_j}\right) = \operatorname{st}\left(\frac{\delta_{m+1}'}{\gamma_j f}\right)$. In this case the criteria are satisfied and the back-and-forth step is again complete.

If $\{\gamma_j, e\}$ is not strongly independent then by proposition 8.2.7 we have that $\gamma_j - qe = c$ for some $q \in \mathbb{Q}$ and some c with $\mathrm{v}(c) < \mathrm{v}(e)$. Again, let $k \in \mathbb{N}$ be the value for which $\gamma'_k = \beta(f(\mathrm{v}(\gamma_j)))$. Then set $\delta_m = qe$ so that $\delta_m \frown \gamma_j$ and set δ'_m to be an element close to γ'_k for which $\widetilde{\varrho}(\delta'_m) = \widetilde{\varrho}(\delta_m)$. Such an element exists in $\widetilde{\Gamma}$ by lemma 9.3.7.

It is clear that at each of the stages above the back-and-forth criteria are satisfied and that either $e_i \in \langle \delta_1, \dots, \delta_m \rangle$ or $e_i \in \langle \delta_1, \dots, \delta_{m+1} \rangle$ depending on the particular case. We may therefore continue this procedure: the 'back' part being practically identical to the 'forth' part as just stated. We are then able to construct an automorphism g in the usual way. It remains to check that this g is suitable. Clearly $g\colon a_i \mapsto b_i$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$. As stated at the outset, we also require that $\gamma_i g \frown \beta(f(\mathrm{v}(\gamma_i)))$ for all $i \in \mathbb{N}$. But by criterion 3 we know that for every γ_i there is some $\delta_k \frown \gamma_i$ for which $g\colon \delta_k \mapsto \delta'_k$ and $\delta'_k \frown \beta(f(\mathrm{v}(\gamma_i)))$. But then

$$\begin{aligned} \mathrm{st}\left(\frac{\gamma_i g}{\beta(f(\mathrm{v}(\gamma_i)))}\right) &= \mathrm{st}\left(\frac{\gamma_i g}{\delta_k g}\right) \cdot \mathrm{st}\left(\frac{\delta_k g}{\beta(f(\mathrm{v}(\gamma_i)))}\right) \\ &= \mathrm{st}\left(\frac{\gamma_i}{\delta_k}\right) \cdot \mathrm{st}\left(\frac{\delta'_k}{\beta(f(\mathrm{v}(\gamma_i)))}\right) \\ &= 1 \end{aligned}$$

which gives us the required result. □

Lemma 10.4.3. Suppose $\mathrm{Res}(\Gamma) \neq \mathbb{Z}$ and $h \in \widetilde{G}_{\mathrm{v}}$ fixes no initial segments. Then for all $s \in \widetilde{\Gamma} \setminus \{0\}$ we can find some $c \in \widetilde{\Gamma}$ with $\mathrm{v}(c) < \mathrm{v}(s)$ and $\widetilde{\varrho}(c) = 0$, and so that if $\mathrm{st}\left(\frac{ch}{c}\right) = q \in \mathbb{Q}$ then $ch \neq qc$.

Proof. Since $\mathrm{Res}(\Gamma) \neq \mathbb{Z}$ it follows by p.r.s.(1) that we can find some $c' \in \widetilde{\Gamma}$ with $\mathrm{v}(c') < \mathrm{v}(s)$ and with non-zero but otherwise arbitrary residue, so that if $\mathrm{st}\left(\frac{c'h}{c'}\right) = q$ then $c'h \neq qc'$. If there is some such c' so that $q \neq 1$ then this is straightforward, since if $\widetilde{\varrho}(c') \neq 0$ we know that $\widetilde{\varrho}(qc') \neq \widetilde{\varrho}(c') = \widetilde{\varrho}(c'h)$ and hence that $qc' \neq c'h$.

However it may be the case that $\mathrm{st}\left(\frac{\gamma h}{\gamma}\right) = 1$ for all γ with $\mathrm{v}(\gamma) < \mathrm{v}(s)$. But the fact that h fixes no initial segments means that we can find some element c' with $\mathrm{v}(c') < \mathrm{v}(s)$ and for which $c'h \neq qc'$ (where $q = 1$).

So in both cases we have either that $qc' < c'h$ or $qc' > c'h$. If $qc' < c'h$ then by p.r.s.(1) we can find some element $c \in \widetilde{\Gamma}$ with $\widetilde{\varrho}(c) = 0$ and $qc' < c < c'h$. Then $qc'h < ch$ and $qc < qc'h$, hence $qc < ch$ and in particular we know that $qc \neq ch$. The case when $qc' > c'h$ follows analogously. □

We will see in section 11.2 later that the requirement in the previous theorem that $\mathrm{Res}(\Gamma) \neq \mathbb{Z}$ is equivalent to saying that $\widetilde{G}$ has trivial centre.

In the next result we will go some way towards our goal of showing how conjugation can affect value-preserving automorphisms. However, it is restricted to tuples of elements with distinct values; a restriction which ideally we would hope to remove. In fact the theorem tells us that we can map elements 'close to' where we might want them, but not necessarily to a precise position. In this respect the restriction to tuples with distinct values can be seen as being irrelevant, but of course the hope is to be able to map to precise elements whilst also ignoring the distinction of valuation classes.

Proposition 10.4.4. Let $h \in \widetilde{G}_{\mathrm{v}}$ and suppose that $a_1, \ldots, a_n \in \widetilde{\Gamma}$ are such that $0 < \mathrm{v}(a_n) < \mathrm{v}(a_{n-1}) < \cdots < \mathrm{v}(a_1)$. Suppose further that $b_1, \ldots, b_n \in \widetilde{\Gamma}$ are such that $\mathrm{tp}(a_i) = \mathrm{tp}(b_i)$ with $\mathrm{st}\left(\frac{b_i}{a_i}\right) \in \mathrm{stQ}(h)$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$. Then there exists some $w \in \left\langle h^{\widetilde{G}} \right\rangle$ such that $a_i w \frown b_i$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$.

We can consider this proposition in terms of the projections of the space $\mathrm{stQ}^{\mathrm{V}}$. It tells us that for $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ elements of V, if there are projections $\pi_1, \ldots, \pi_n \colon \mathrm{stQ}^{\mathrm{V}} \to (\mathrm{stQ}(\Gamma))^1$ with $\mathrm{st}\left(\frac{b_i}{a_i}\right) = \pi_i(\theta(h))$ then there is some $w \in \left\langle h^{\widetilde{G}} \right\rangle$ and a projection $\pi \colon \mathrm{stQ}^{\mathrm{V}} \to (\mathrm{stQ}(\Gamma))^n$ such that

$$\pi(\theta(w)) = \left(\mathrm{st}\left(\frac{b_1}{a_1}\right), \ldots, \mathrm{st}\left(\frac{b_n}{a_n}\right)\right).$$

At this stage, however, it's worth pointing out that the proposition does actually tell us a little more than this. For the description in terms of projections will only hold if there are $\gamma_1, \ldots, \gamma_n$ with $\mathrm{v}(\gamma_1) < \cdots < \mathrm{v}(\gamma_n)$ and such that

$$\left(\mathrm{st}\left(\frac{\gamma_1 h}{\gamma_1}\right), \ldots, \mathrm{st}\left(\frac{\gamma_n h}{\gamma_n}\right)\right) = \left(\mathrm{st}\left(\frac{b_1}{a_1}\right), \ldots, \mathrm{st}\left(\frac{b_n}{a_n}\right)\right),$$

whilst the proposition will hold for any tuple

$$\left(\mathrm{st}\left(\frac{b_1}{a_1}\right), \ldots, \mathrm{st}\left(\frac{b_n}{a_n}\right)\right) \in (\mathrm{stQ}(\Gamma))^n.$$

So in the proposition the ordering of the standard parts is irrelevant and we are also entitled to include duplicates of standard parts, whilst when considering projections these properties become relevant. Although we will use this extra flexibility of the proposition, we will see later that it is actually not necessary to do so, as any results derived in this way will follow naturally from our later conclusions.

Proof. We prove this by induction on n.

For our inductive hypothesis we suppose that for $a_1, \dots, a_m$ and $b_1, \dots, b_m$ as given in the statement we can find some word w for which $a_i w \frown b_i$ for $i \in \mathbb{N}$ with $1 \leq i \leq m$. We also require that the set

$$\{\mathrm{v}(\gamma) : \gamma \in \widetilde{\Gamma}, 0 < \gamma \leq a_m\},$$

when considered as a coloured set of ordered values determined by w, is to be isomorphic to an initial interval

$$\{\mathrm{v}(\gamma) : \gamma \in \widetilde{\Gamma}, 0 < \gamma \leq v_m\}$$

of the coloured set of ordered values determined by h for some $v_m \in \widetilde{\Gamma}$.

For the base case we must ensure that $a_1 w \frown b_1$. We know that $\mathrm{st}\left(\frac{b_1}{a_1}\right) = r_1$ for some $r_1 \in \mathrm{stQ}(h)$, so we know that for some $v_1' \in \widetilde{\Gamma}$ it is the case that $\mathrm{st}\left(\frac{v_1' h}{v_1'}\right) = r_1$. By lemma 9.3.7 we can find some v_1 close to v_1' for which $\widetilde{\varrho}(v_1) = \widetilde{\varrho}(a_1)$. Then $\mathrm{tp}(a_1) = \mathrm{tp}(v_1)$. We intend to construct a map $g\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ for which $g\colon v_1 \mapsto a_1$.

Let $W_{v_1} = \{\, \mathrm{v}(\gamma)\colon 0 < \mathrm{v}(\gamma) \leq \mathrm{v}(v_1)\}$ and $W_{a_1} = \{\, \mathrm{v}(\gamma)\colon 0 < \mathrm{v}(\gamma) \leq \mathrm{v}(a_1)\}$. Clearly W_{v_1} and W_{a_1} are isomorphic when considered as dense linear orders, so we may let $f\colon W_{v_1} \to W_{a_1}$ be an isomorphism between them. By proposition 10.4.2 we can construct a residue automorphism g which maps v_1 to a_1 and such that $\mathrm{v}(\gamma g) = f(\mathrm{v}(\gamma))$ for all $\gamma < v_1$. We claim that the word $w = g^{-1} h g$ will then do. Now

$$\begin{aligned} \mathrm{st}\left(\frac{a_1 w}{a_1}\right) &= \mathrm{st}\left(\frac{a_1 g^{-1} h}{a_1 g^{-1}}\right) \\ &= \mathrm{st}\left(\frac{v_1 h}{v_1}\right) \\ &= r_1 \\ &= \mathrm{st}\left(\frac{b_1}{a_1}\right) \end{aligned}$$

and so by lemma 8.2.13 we know that $a_1 g \frown b_1$ as required. It is clear that g^{-1} when considered as a mapping of the values $\{\, \mathrm{v}(\gamma)\colon \mathrm{v}(\gamma) \leq \mathrm{v}(a_1)\}$ to the values $\{\, \mathrm{v}(\gamma)\colon \mathrm{v}(\gamma) \leq \mathrm{v}(v_1)\}$ of $\widetilde{\Gamma}$ acts as an isomorphism of dense linear orders. We also claim that this is an isomorphism when the sets are considered as coloured sets of ordered values, *i.e.* that

$$\mathrm{st}\left(\frac{\gamma w}{\gamma}\right) = \mathrm{st}\left(\frac{\gamma g^{-1} h}{\gamma g^{-1}}\right).$$

But this is trivially true and so w will satisfy the inductive hypothesis for the base case.

We must now consider the inductive step. Suppose we have $a_1, \ldots, a_{m-1}$ and $b_1, \ldots, b_{m-1}$ and some word $w \in \left\langle h^{\widetilde{G}} \right\rangle$ which satisfy the inductive hypothesis. Suppose further that $a_m, b_m \in \widetilde{\Gamma}$ are such that $\mathrm{v}(a_m) < \mathrm{v}(a_{m-1})$ and that $\mathrm{st}\left(\frac{b_m}{a_m}\right) = r_m$ for some $r_m \in \mathrm{stQ}(h)$. We will now construct a pair of automorphisms $g_1, g_2 \in G$ with the intention of showing that if $w' = g_1^{-1}h^{-1}g_1g_2^{-1}hg_2$ then the word ww' will suffice.

By the inductive hypothesis we know that for some $v_{m-1} \in \widetilde{\Gamma}$ it is the case that the set of values

$$W_{r_{m-1}} = \{\, \mathrm{v}(\gamma) : \gamma \in \widetilde{\Gamma}, 0 < \gamma \leq v_{m-1}\}$$

when considered as a coloured set of ordered values determined by h is isomorphic to the initial interval

$$W_{a_{m-1}} = \{\, \mathrm{v}(\gamma) : \gamma \in \widetilde{\Gamma}, 0 < \gamma \leq a_{m-1}\}$$

of the coloured set of ordered values determined by w. Let c_h and c_w be the respective colouring maps and let $f' \colon W_{r_{m-1}} \to W_{a_{m-1}}$ be an isomorphism between the two sets. Since $\mathrm{v}(a_m) < \mathrm{v}(a_{m-1})$ we know that $\mathrm{v}(a_m)$ is contained in the second of these sets and hence that $f'^{-1}(\mathrm{v}(a_m))$ exists.

We also know that for some $v'_m \in \widetilde{\Gamma}$ we have $\mathrm{st}\left(\frac{b_m}{a_m}\right) = \mathrm{st}\left(\frac{v'_m h}{v'_m}\right)$. By lemma 9.3.7 we may find some $v_m \frown v'_m$ and so that $\widetilde{\varrho}(v_m) = \widetilde{\varrho}(a_m)$. Choose $s_1, \ldots, s_{m-1}$ so that $\mathrm{v}(a_m), \mathrm{v}(v_m) < \mathrm{v}(s_{m-1}) < \cdots < \mathrm{v}(s_1)$ and with $\widetilde{\varrho}(s_i) = \widetilde{\varrho}(a_i)$ for all $i \in \mathbb{N}$ with $1 \leq i \leq m-1$.

Now we construct g_1 using proposition 10.4.2 so that $g_1 \colon s_i \mapsto a_i w$ for all $i \in \mathbb{N}$ such that $1 \leq i \leq m-1$ and with $\mathrm{v}(\gamma g_1) = f'(\mathrm{v}(\gamma))$ for all $\gamma \leq a_m$. But f' represents an isomorphism of the coloured sets of ordered values as defined above, and hence $c_w(\mathrm{v}(\gamma)) = c_h(f'^{-1}(\mathrm{v}(\gamma)))$ for all $\gamma \leq a_m$. We therefore see that $c_w(\mathrm{v}(\gamma)) = c_h(\mathrm{v}(\gamma g_1^{-1}))$. In other words

$$\mathrm{st}\left(\frac{\gamma w}{\gamma}\right) = \mathrm{st}\left(\frac{\gamma g_1^{-1} h}{\gamma g_1^{-1}}\right),$$

and hence

$$\begin{aligned} \mathrm{st}\left(\frac{\gamma w g_1^{-1} h^{-1} g_1}{\gamma}\right) &= \mathrm{st}\left(\frac{\gamma w}{\gamma g_1^{-1} h g_1}\right) \\ &= \mathrm{st}\left(\frac{\gamma w}{\gamma}\right) \cdot \mathrm{st}\left(\frac{\gamma}{\gamma g_1^{-1} h g_1}\right) \\ &= 1, \end{aligned}$$

and so by lemma 10.1.3 we have that $\gamma w g_1^{-1} h^{-1} g_1 \frown \gamma$ for all $\gamma \leq a_m$ (note that this includes a_m).

We now construct g_2 and again we use proposition 10.4.2. So let $g_2 \colon s_i h^{-1} \mapsto a_i w g_1^{-1} h^{-1} g_1$ for all $i \in \mathbb{N}$ with $1 \leq i \leq m-1$ and with $g_2 \colon v_m \mapsto a_m$. Our choice of s_i ensures that we can do this. Proposition 10.4.2 also allows us to ensure that the set of values

$$\{\mathrm{v}(\gamma) : \gamma \in \widetilde{\Gamma}, 0 < \gamma \leq a_m\}$$

when considered as a coloured set of ordered values determined by $g_2^{-1} h g_2$ is isomorphic to the initial interval

$$\{\mathrm{v}(\gamma) : \gamma \in \widetilde{\Gamma}, 0 < \gamma \leq v_m\}$$

of the coloured set of ordered values determined by h.

Now for $i \in \mathbb{N}$ with $1 \leq i \leq m-1$ we know that $a_i w \frown b_i$ and so by using lemma 10.1.3 and lemma 10.4.1 we have that

$$\begin{aligned}
\mathrm{st}\left(\frac{a_i w w'}{a_i}\right) &= \mathrm{st}\left(\frac{a_i w g_1^{-1} h^{-1} g_1 g_2^{-1} h g_2}{a_i}\right) \\
&= \mathrm{st}\left(\frac{a_i w g_1^{-1} h^{-1} g_1 g_2^{-1} h g_2}{a_i w g_1^{-1} h^{-1} g_1}\right) \cdot \mathrm{st}\left(\frac{a_i w g_1^{-1} h^{-1} g_1}{a_i}\right) \\
&= \mathrm{st}\left(\frac{a_i g_2^{-1} h g_2}{a_i}\right) \cdot \mathrm{st}\left(\frac{a_i w g_1^{-1} h^{-1} g_1}{a_i w}\right) \cdot \mathrm{st}\left(\frac{a_i w}{a_i}\right) \\
&= \mathrm{st}\left(\frac{a_i g_2^{-1} h}{a_i g_2^{-1}}\right) \cdot \mathrm{st}\left(\frac{s_i h^{-1} g_1}{s_i g_1}\right) \cdot \mathrm{st}\left(\frac{a_i w}{a_i}\right) \\
&= \mathrm{st}\left(\frac{s_i h}{s_i}\right) \cdot \mathrm{st}\left(\frac{s_i h^{-1}}{s_i}\right) \cdot \mathrm{st}\left(\frac{b_i}{a_i}\right) \\
&= \mathrm{st}\left(\frac{b_i}{a_i}\right).
\end{aligned}$$

We also have that

$$\begin{aligned}
\mathrm{st}\left(\frac{a_m w w'}{a_m}\right) &= \mathrm{st}\left(\frac{a_m w g_1^{-1} h^{-1} g_1 g_2^{-1} h g_2}{a_m w g_1^{-1} h^{-1} g_1}\right) \cdot \mathrm{st}\left(\frac{a_m w g_1^{-1} h^{-1} g_1}{a_m}\right) \\
&= \mathrm{st}\left(\frac{a_m g_2^{-1} h g_2}{a_m}\right) \cdot 1 \\
&= \mathrm{st}\left(\frac{v_m h}{v_m}\right) = r_m
\end{aligned}$$

and so $a_m w w' \frown b_m$ as required. The remainder of the inductive hypothesis follows from the construction. □

The next proposition is a technical result which we will use later. In the form stated here it does not tell us anything particularly enlightening about conjugates of value-preserving automorphisms, however it is given here since its use will nonetheless be restricted to these. Its purpose is to show that, given certain assumptions and certain automorphisms, we have a great deal of flexibility when it comes to mapping strongly independent elements which are close to each other, using conjugates.

Proposition 10.4.5. Let Γ be a pseudo-recursively saturated model of Presburger arithmetic and let $h \in \widetilde{G}_{\mathrm{v}}$. Suppose we have $a_1, \dots, a_n \in \widetilde{\Gamma}$ strongly independent and that for some fixed k we have $b_k \frown a_k$ with $\widetilde{\varrho}(a_k) = \widetilde{\varrho}(b_k)$ and $\mathrm{v}(a_k) \leq \mathrm{v}(a_i)$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$. Suppose further that $s_1, \dots, s_n \in \widetilde{\Gamma}$ are such that $\widetilde{\varrho}(s_i) = \widetilde{\varrho}(a_i)$ and $\mathrm{st}\left(\frac{s_i}{s_j}\right) = \mathrm{st}\left(\frac{a_i}{a_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$, and that $d_1, \dots, d_n \in \widetilde{\Gamma}$ are strongly independent with $\mathrm{v}(d_i) < \mathrm{v}(s_j)$ for all $i, j \in \mathbb{N}$ such that $1 \leq i, j \leq n$. Additionally, suppose that

$$s_i h \in \langle s_1, \dots, s_i, d_1, \dots, d_i \rangle$$

whilst

$$s_i h \notin \langle s_1, \dots, s_i, d_1, \dots, d_{i-1} \rangle \cup \langle s_1, \dots, s_{i-1}, d_1, \dots, d_i \rangle \tag{10.1}$$

for $i \in \mathbb{N}$ with $1 \leq i \leq n$.

Then there exist $d'_1, \dots, d'_n \in \widetilde{\Gamma}$ so that $\widetilde{\varrho}(d'_i) = \widetilde{\varrho}(d_i)$ and $\mathrm{st}\left(\frac{d'_i}{d'_j}\right) = \mathrm{st}\left(\frac{d_i}{d_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$. Moreover there also exist $c'_k, \dots, c'_n \in \widetilde{\Gamma}$ with $\mathrm{v}(c'_j) < \mathrm{v}(d'_i)$ and $\widetilde{\varrho}(c'_j) = 0$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$ and $k \in \mathbb{N}$ with $1 \leq k \leq n$ such that for any $g_1, g_2 \in \widetilde{G}$ with

$$g_1\colon \begin{array}{rcll} s_1 & \mapsto & a_1 & ; \\ \vdots & & \vdots & \\ s_n & \mapsto & a_n & ; \\ d_1 & \mapsto & d'_1 & ; \\ \vdots & & \vdots & \\ d_n & \mapsto & d'_n & \end{array} \qquad \text{and} \qquad g_2\colon \begin{array}{rcll} s_1 h & \mapsto & s_1 h g_1 & ; \\ \vdots & & \vdots & \\ s_n h & \mapsto & s_n h g_1 & ; \\ d_1 & \mapsto & d'_1 & ; \\ \vdots & & \vdots & \\ d_{k-1} & \mapsto & d'_{k-1} & ; \\ d_k & \mapsto & d'_k + c'_k & ; \\ \vdots & & \vdots & \\ d_n & \mapsto & d'_n + c'_n & , \end{array}$$

we have

$$\begin{aligned} a_i w &= a_i \quad \text{for } i \neq k, \\ a_k w &= b_k \end{aligned}$$

where $w = g_1^{-1} h g_1 g_2^{-1} h^{-1} g_2$ and where instances of g_1 and g_2 can be shown to exist.

Proof. We begin by noting that, referring ahead to section 11.2, the requirements of the theorem ensure that h is non-central and indeed non-trivial. If h were central then for every $\gamma \in \widetilde{\Gamma}$ we would have $\gamma h = q\gamma$ for some $q \in \mathbb{Q}$ and hence $\gamma h \in \langle \gamma \rangle$. The requirement (10.1) would clearly be contradicted if this were the case.

For the proof proper, since $a_k \frown b_k$ we know by proposition 8.2.7 that $a_k - b_k = c$ for some c with $\mathrm{v}(c) < \mathrm{v}(a_k) = \mathrm{v}(b_k) \leq \mathrm{v}(a_i)$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$. We choose $d_1', \ldots, d_n'$ so that $\mathrm{v}(c) < \mathrm{v}(d_i') < \mathrm{v}(a_j)$, so that $\widetilde{\varrho}(d_i) = \widetilde{\varrho}(d_i')$ and with $\mathrm{st}\left(\frac{d_i}{d_j}\right) = \mathrm{st}\left(\frac{d_i'}{d_j'}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$. We can do this since $\widetilde{\Gamma}$ is pseudo-recursively saturated.

For each s_i we know by the assumptions that $s_i h \in \langle s_1, d_1, \ldots, s_i, d_i \rangle$. We therefore have that

$$s_i h = q_1 s_1 + \cdots + q_i s_i + q_1' d_1 + \cdots + q_i' d_i$$

for some $q_1, \ldots, q_i, q_1', \ldots, q_i' \in \mathbb{Q}$ (which depend upon i). It follows that

$$s_i = q_i^{-1}(s_i h - (q_1 s_1 + \cdots + q_{i-1} s_{i-1} + q_1' d_1 + \cdots + q_i' d_i)),$$

since $q_i \neq 0$ by (10.1).

We define the elements $c_k', \ldots, c_n'$ inductively as follows: Suppose we have defined $c_k', \ldots, c_{i-1}'$ and wish to define c_i' and that, following the discussion above,

$$s_i h = q_1 s_1 + \cdots + q_i s_i + q_1' d_1 + \cdots + q_i' d_i$$

for some $q_1, \ldots, q_i, q_1', \ldots, q_i' \in \mathbb{Q}$. Then we define c_i' by setting

$$c_i' = \frac{q_k c - q_k' c_k' - \ldots - q_{i-1}' c_{i-1}'}{q_i'}$$

where the base case, $i = k$, is defined by the same rule as

$$c_k' = \frac{q_k c}{q_k'}.$$

We can do this since in each case we have $q_i, q_i' \neq 0$ by (10.1) above. Again, note that in each instance the $q_1, \ldots, q_n, q_1', \ldots, q_n'$ will differ.

Now for every $i \in \mathbb{N}$ with $1 \leq i \leq n$ we have $c_i' = qc$ for some $q \in \mathbb{Q}$ and since $\mathrm{v}(c) < \mathrm{v}(d_i')$ we therefore know that $d_i' + c_i' \frown d_i'$. We also have that $\widetilde{\varrho}(c) = 0$, ensuring that $\widetilde{\varrho}(d_i' + c_i') = \widetilde{\varrho}(d_i')$ over the same i. It follows that the pairs of sets

$$\{s_1, \ldots, s_n, d_1, \ldots, d_n\},$$

$$\{a_1, \ldots, a_n, d_1', \ldots, d_n'\},$$

and

$$\{s_1h, \ldots, s_nh, d_1, \ldots, d_n\},$$
$$\{s_1hg_1, \ldots, s_nhg_1, d'_1, \ldots, d'_{k-1}, d'_k + c'_k, \ldots, d'_n + c'_n\},$$

have consistent standard parts and residues and are strongly independent by lemma 8.2.8, hence it is certainly possible to construct residue automorphisms $g_1 \colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ and $g_2 \colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ so that

$$g_1\colon \begin{array}{ccc} s_1 & \mapsto & a_1 \quad ; \\ \vdots & & \vdots \\ s_n & \mapsto & a_n \quad ; \\ d_1 & \mapsto & d'_1 \quad ; \\ \vdots & & \vdots \\ d_n & \mapsto & d'_n \quad , \end{array} \qquad g_2\colon \begin{array}{ccc} s_1h & \mapsto & s_1hg_1 \quad ; \\ \vdots & & \vdots \\ s_nh & \mapsto & s_nhg_1 \quad ; \\ d_1 & \mapsto & d'_1 \quad ; \\ \vdots & & \vdots \\ d_{k-1} & \mapsto & d'_{k-1} \quad ; \\ d_k & \mapsto & d'_k + c'_k \quad ; \\ \vdots & & \vdots \\ d_n & \mapsto & d'_n + c'_n \quad . \end{array}$$

We must show that for any residue automorphisms which map as g_1 and g_2 do, the word $w = g_1^{-1}hg_1g_2^{-1}h^{-1}g_2$ will act as expected. We note that for an element a_i where $i \in \mathbb{N}$ and $1 \leq i \leq n$ we have

$$\begin{aligned} a_iw &= a_ig_1^{-1}hg_1g_2^{-1}h^{-1}g_2 \\ &= s_ihg_1g_2^{-1}h^{-1}g_2 \\ &= s_ig_2. \end{aligned}$$

Now we begin by showing that $a_iw = a_i$ for $i \in \mathbb{N}$ with $1 \leq i < k$. We do this by induction on i. So for the base step we consider a_1. We know that $a_1w = s_1g_2$ and that $s_1h = q_1s_1 + q'_1d_1$ for some $q_1, q'_1 \in \mathbb{Q}$. So

$$\begin{aligned} s_1g_2 &= q_1^{-1}(s_1hg_2 - q'_1d_1g_2) \\ &= q_1^{-1}(s_1hg_1 - q'_1d'_1) \\ &= q_1^{-1}(s_1h - q'_1d_1)g_1 \\ &= s_1g_1 = a_1. \end{aligned}$$

Hence $a_1 w = a_1$ as required. Now suppose inductively that $s_i g_2 = a_i$ for all $i < m$ where $m < k$. We wish to show that $s_m g_2 = a_m$ and hence that $a_m w = a_m$. Again, we have that

$$s_m h = q_1 s_1 + \cdots + q_m s_m + q'_1 d_1 + \cdots + q'_m d_m$$

for some $q_1, \ldots, q_m, q'_1, \ldots, q'_m \in \mathbb{Q}$ (note that we take these to be unconnected to previous occurrences of q_j or q'_j for $j \in \mathbb{N}$). Hence

$$s_m = q_m^{-1}(s_m h - (q_1 s_1 + \cdots + q_{m-1} s_{m-1} + q'_1 d_1 + \cdots + q'_m d_m)).$$

Then

$$\begin{aligned} s_m g_2 &= q_m^{-1}(s_m h g_2 - (q_1 s_1 g_2 + \cdots + q_{m-1} s_{m-1} g_2 + q'_1 d_1 g_2 + \cdots + q'_m d_m g_2)) \\ &= q_m^{-1}(s_m h g_1 - (q_1 a_1 + \cdots + q_{m-1} a_{m-1} + q'_1 d'_1 + \cdots + q'_m d'_m)) \\ &= q_m^{-1}(q_1 s_1 g_1 + \cdots + q_m s_m g_1 + q'_1 d_1 g_1 + \cdots + q'_m d_m g_1 \\ &\qquad - (q_1 a_1 + \cdots + q_{m-1} a_{m-1} + q'_1 d'_1 + \cdots + q'_m d'_m)) \\ &= q_m^{-1}(q_m s_m g_1) = a_m \end{aligned}$$

as required. This completes the inductive step.

We now consider the element a_k and claim that $a_k w = b_k$. We have that

$$s_k h = q_1 s_1 + \cdots + q_k s_k + q'_1 d_1 + \cdots + q'_k d_k$$

Hence

$$s_k = q_k^{-1}(s_k h - (q_1 s_1 + \cdots + q_{k-1} s_{k-1} + q'_1 d_1 + \cdots + q'_i d_k)).$$

So

$$\begin{aligned} s_k g_2 &= q_k^{-1}(s_k h g_2 - (q_1 s_1 g_2 + \cdots + q_{k-1} s_{k-1} g_2 + q'_1 d_1 g_2 + \cdots + q'_k d_k g_2)) \\ &= q_k^{-1}(s_k h g_1 - (q_1 a_1 + \cdots + q_{k-1} a_{k-1} + q'_1 d'_1 + \cdots + q'_{k-1} d'_{k-1} + q'_k(d'_k + c'_k))) \\ &= q_k^{-1}(q_1 s_1 g_1 + \cdots + q_k s_k g_1 + q'_1 d_1 g_1 + \cdots + q'_k d_k g_1 \\ &\qquad - (q_1 a_1 + \cdots + q_{k-1} a_{k-1} + q'_1 d'_1 + \cdots + q'_{k-1} d'_{k-1} + q'_k(d'_k + c'_k))) \\ &= q_k^{-1}(q_k s_k g_1 + q'_k d'_k - (q'_k(d'_k + c'_k))) \\ &= q_k^{-1}(q_k a_k - q'_k c'_k) \\ &= a_k - \frac{q'_k}{q_k} c'_k \\ &= b_k. \end{aligned}$$

So $a_k w = s_k g_2 = b_k$ as required. It remains to show that $a_i w = a_i$ for $k < j \leq n$. Again we prove this inductively. So suppose that $s_i g_2 = a_i$ for all $k < i < m$ where $k < m \leq n$. Then

$$s_m h = q_1 s_1 + \cdots + q_m s_m + q_1' d_1 + \cdots + q_m' d_m$$

for some $q_1, \ldots, q_m, q_1', \ldots, q_m' \in \mathbb{Q}$ (where again we take these to be unconnected to previous occurrences of q_j or q_j' for $j \in \mathbb{N}$). Hence

$$s_m = q_m^{-1}(s_m h - (q_1 s_1 + \cdots + q_{m-1} s_{m-1} + q_1' d_1 + \cdots + q_m' d_m)).$$

Then

$$\begin{aligned}
s_m g_2 &= q_m^{-1}(s_m h g_2 - (q_1 s_1 g_2 + \cdots + q_{m-1} s_{m-1} g_2 + q_1' d_1 g_2 + \cdots + q_m' d_m g_2)) \\
&= q_m^{-1}(s_m h g_1 - (q_1 a_1 + \cdots + q_{k-1} a_{k-1} + q_k b_k + q_{k+1} a_{k+1} + \cdots + q_{m-1} a_{m-1} \\
&\qquad + q_1' d_1' + \cdots + q_{k-1}' d_{k-1}' + q_k'(d_k' + c_k') + \cdots + q_m'(d_m' + c_m') \\
&= q_m^{-1}(q_1 a_1 + \cdots + q_m a_m + q_1' d_1' + \cdots + q_m' d_m' \\
&\qquad - (q_1 a_1 + \cdots + q_{k-1} a_{k-1} + q_k b_k + q_{k+1} a_{k+1} + \cdots + q_{m-1} a_{m-1} \\
&\qquad\qquad + q_1' d_1' + \cdots + q_{k-1}' d_{k-1}' + q_k'(d_k' + c_k') + \cdots + q_m'(d_m' + c_m') \\
&= q_m^{-1}(q_k a_k + q_m a_m + q_k' d_k' + \cdots + q_m' d_m' \\
&\qquad - (q_k b_k + q_k'(d_k' + c_k') + \cdots + q_m'(d_m' + c_m'))) \\
&= a_m + q_m^{-1}(q_k c + q_k'(d_k' - d_k' - c_k') + \cdots + q_m'(d_m' - d_m' - c_m')) \\
&= a_m + q_m^{-1}(q_k c - q_k' c_k' - \cdots - q_m' c_m'),
\end{aligned}$$

but by the definition of c_m' we have that $q_m' c_m' = q_k c - q_k' c_k' - \cdots - q_{m-1}' c_{m-1}'$ and hence

$$s_m g_2 = a_m.$$

But then $a_m w = s_m g_2 = a_m$ as required. $\square$

The full results which are needed in order to fully understand conjugates of value-preserving automorphisms will be given later on when we consider the structure of the closed normal subgroups of G. Although this will be done in the next chapter, we leave it to the latter half, looking in the first half at some examples which will help to elucidate the form of the groups which we will be looking for.

Chapter 11

The Normal Subgroups of $\mathrm{Aut}(\Gamma)$

11.1 The topology of $\mathrm{Aut}(\Gamma)$

The aim of this chapter is to exhibit the closed normal subgroups of the automorphism group of a countable pseudo-recursively saturated model of Presburger arithmetic. As in the previous chapter we shall refer to this group $\mathrm{Aut}(\Gamma)$ simply as G and may also use $\widetilde{G}$ to mean $G/\mathbb{Z}$. We apply a topology to G by making a basic open set in G to be a set $G_{(S)}$ where this is defined in the following way:

Definition 11.1.1. If $S \subseteq \Gamma$ is finite then we define

$$G_{(S)} = \{\, g \in G : xg = x \text{ for all } x \in S\}.$$

These basic open sets form a neighbourhood base of the identity for the topology. We will be restricting our investigation to the normal subgroups which are closed with respect to this topology, a move for which there is a specific model-theoretic motivation. A further brief discussion of this is given in the section 12.1 on orbits, but for more general information about this natural topology, see Kaye and Macpherson [32, pp. 18 ff.].

11.2 The centre of G

An obvious example of a normal subgroup of G which springs to mind is its centre. However, if Γ is a pseudo-recursively saturated model of Presburger arithmetic with $\mathrm{Res}(\Gamma) \neq \mathbb{Z}$ then the centre turns out to be trivial, as the next three results show.

In the case when $\mathrm{Res}(\Gamma) = \mathbb{Z}$ this does not turn out to be the case, and it will be interesting therefore to digress slightly in order to consider this case separately.

Lemma 11.2.1. Suppose Γ is a model of Presburger arithmetic and $g \in \mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})$. Then for $x \in \widetilde{\Gamma}$, $q \in \mathbb{Q} \setminus \{1\}$,

$$xg = qx \quad \Rightarrow \quad \widetilde{\varrho}(x) = 0.$$

Proof. Suppose that $xg = qx$ for some $q \in \mathbb{Q} \setminus \{1\}$. Then $\widetilde{\varrho}(xg) = \widetilde{\varrho}(qx)$. Hence $\widetilde{\varrho}(x) = q\widetilde{\varrho}(x)$ and so $s\widetilde{\varrho}(x) = r\widetilde{\varrho}(x)$ for some $r, s \in \mathbb{Z}$ with $r \neq s$. Hence $(r-s)\widetilde{\varrho}(x) = 0$. But $(r - s) \neq 0$ so we must have $\widetilde{\varrho}(x) = 0$ as required. □

Lemma 11.2.2. Suppose Γ is a pseudo-recursively saturated model of Presburger arithmetic with $\mathrm{Res}(\Gamma) \neq \mathbb{Z}$ and $g \in \mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})$, $g \neq 1$. Then there exist $x, y \in \widetilde{\Gamma}$ with $xg = y$ and so that $x \notin \mathbb{Q} \cdot y = \{\, \gamma \in \widetilde{\Gamma} : \gamma = qy, q \in \mathbb{Q}\}$.

Proof. Since $g \neq 1$ we can find some $\gamma \in \Gamma$ for which $\gamma g \neq \gamma$.

If $\gamma g \neq q\gamma$ for any $q \in \mathbb{Q}$ then we're done. So we may suppose that $\gamma g = q'\gamma$ for some $q' \in \mathbb{Q}$. Clearly $q' \neq 1$ as this would contradict the fact that $\gamma g \neq \gamma$, so by the previous lemma $\widetilde{\varrho}(\gamma) = 0$.

Now suppose that $\gamma < \gamma g$. Since $\mathrm{Res}(\Gamma) \neq \mathbb{Z}$ we can use p.r.s.(1) to find some x such that $\gamma < x < \gamma g$ with $\widetilde{\varrho}(x) \neq 0$. Now $\gamma < x$ so $\gamma g < xg$ and hence $x \neq xg$. Equivalently we can write this as $xg \neq 1 \cdot x$. Now applying the previous lemma again we see that $xg \neq qx$ for any $q \in \mathbb{Q} \setminus \{1\}$ since $\widetilde{\varrho}(x) \neq 0$. This x will clearly suffice.

On the other hand, if $\gamma > \gamma g$ then an almost identical argument holds, with limits reversed throughout. □

Theorem 11.2.3. Suppose Γ is a pseudo-recursively saturated model of Presburger arithmetic. Then the centre of $\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})$ is given by

$$Z(\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})) = \left\{ g \in \mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho}) : \exists\, q \in \mathbb{Q}\ \forall\, x \in \widetilde{\Gamma}\ xg = qx \right\}.$$

Proof. Let

$$Z_1 = \left\{ g \in \mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho}) : \exists\, q \in \mathbb{Q}\ \forall\, x \in \widetilde{\Gamma}\ xg = qx \right\}$$

and

$$Z_\omega = \left\{ g \in \mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho}) : \forall\, x \in \widetilde{\Gamma}\ \exists\, q \in \mathbb{Q}\ xg = qx \right\}.$$

The fact that

$$Z_1 \subseteq Z(\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho}))$$

is clear, however we also need to show the reverse direction, which is less straightforward. We first show that

$$Z(\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})) \subseteq Z_\omega$$

So suppose $g \notin Z_\omega$. Then $g \neq 1$ and there is some $x \in \widetilde{\Gamma}$ for which $xg \neq qx$ for any $q \in \mathbb{Q}$. In order to show that g is not included in the centre of $\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})$ we want to find some $\alpha \in \mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})$ so that $g\alpha \neq \alpha g$.

To start with, suppose g does not preserve values. Then there is some $x \in \widetilde{\Gamma}$ for which $\mathrm{v}(x) \neq \mathrm{v}(xg)$. In particular then, the set $\{x, xg\}$ is strongly independent. Without loss of generality we may suppose that $x < xg$ (otherwise we simply consider the map g^{-1}). By p.r.s.(1) and p.r.s.(3) we can find some $y \in \widetilde{\Gamma}$ with $\mathrm{v}(xg) < \mathrm{v}(y)$ and so that $\widetilde{\varrho}(x) = \widetilde{\varrho}(xg) = \widetilde{\varrho}(y)$. Then $\{x, y\}$ is also strongly independent, so by theorem 10.1.2 we can construct a map $\alpha \colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ such that

$$\alpha \colon \begin{cases} x & \mapsto x \\ xg & \mapsto y. \end{cases}$$

We then see that $xg\alpha = y$ whilst $x\alpha g = xg$ where $y \neq xg$. So $g\alpha \neq \alpha g$.

We may therefore suppose that g does preserve values. By assumption we may find find some $x, y \in \widetilde{\Gamma}$ with $xg = y$ and $x \notin \mathbb{Q} \cdot y$ (so in particular, note that $x \neq y$). Moreover $\mathrm{v}(x) = \mathrm{v}(y)$, so

$$\mathrm{st}\left(\frac{x}{y}\right) = r \notin \{0, \pm\infty\}.$$

Suppose first that x and y are strongly independent. Choose some $z \in \widetilde{\Gamma}$ with $0 < \mathrm{v}(z) < \mathrm{v}(y)$ and $\widetilde{\varrho}(z) = 0$, which we can do by p.r.s.(1) and p.r.s.(3). If we set $y' = y + z$ then x and y' are strongly independent and have the correct residues and standard parts to allow the use of theorem 10.1.2 in order to construct a residue automorphism $\alpha \colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ such that

$$\alpha \colon \begin{cases} x & \mapsto x \\ y & \mapsto y'. \end{cases}$$

Now $xg\alpha = y\alpha = y'$ whilst $x\alpha g = xg = y$. But $\mathrm{v}(z) \neq 0$ so $y \neq y'$ from which we deduce that $g\alpha \neq \alpha g$.

Suppose now that x and y are not strongly independent. Then by lemma 8.2.5 there is some $q \in \mathbb{Q}$ such that

$$\mathrm{v}(qx + y) < \mathrm{v}(x).$$

We set $\gamma = qx + y$. Since $x \notin \mathbb{Q} \cdot y$ we know that $\gamma \neq 0$.

We claim that we can find some z such that $qz + zg \neq \gamma$ with $\mathrm{v}(z) = \mathrm{v}(x)$ but $z \neq x$ and $\widetilde{\varrho}(z) = \widetilde{\varrho}(x)$. For suppose that such a z did not exist. Then for all z

$$\mathrm{v}(z) = \mathrm{v}(x), z \neq x \text{ and } \widetilde{\varrho}(z) = \widetilde{\varrho}(x) \quad \Rightarrow \quad qz + zg = \gamma. \tag{11.1}$$

By p.r.s.(1) there are infinitely many z satisfying the left hand side of (11.1). Choose some such $z' \neq y$. Then

$$qz'g^{-1} + z' = \gamma g^{-1},$$

which we can re-write as

$$q(z'g^{-1}) + (z'g^{-1})g = \gamma g^{-1}.$$

But g preserves values and $z'g^{-1} \neq x$, so $(z'g^{-1})$ also satisfies (11.1). Hence we may conclude that

$$\gamma = \gamma g.$$

Now $x \neq y$ so suppose $x < y$. Then $y = xg < yg$ so $\gamma = qx + y < qy + yg = \gamma g$ contradicting the fact that $\gamma = \gamma g$. If $x > y$ a similar argument will hold. We conclude therefore that there must exist some z with $\mathrm{v}(z) = \mathrm{v}(x)$, with $\widetilde{\varrho}(z) = \widetilde{\varrho}(x)$ and such that $qz + zg \neq \gamma$. For such a z we set $\gamma' = qz + zg$. Since $\{\gamma, z\}$ and $\{\gamma, x\}$ are both strongly independent sets, we can use lemma 10.1.2 to construct a residue automorphism $\alpha \colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ so that:

$$\alpha \colon \begin{cases} \gamma & \mapsto \gamma; \\ z & \mapsto x. \end{cases}$$

Then $z\alpha g = xg = y = \gamma - qx$ whilst $zg\alpha = (\gamma' - qz)\alpha = \gamma'\alpha - qx$. But now $\gamma\alpha = \gamma$ and $\gamma' \neq \gamma$, hence $\gamma'\alpha \neq \gamma$. It follows that $z\alpha g \neq zg\alpha$.

In no situation do we have $g\alpha = \alpha g$, so $g \notin Z(\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho}))$.

We have shown that $Z(\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})) \subseteq Z_\omega$. We next consider elements $g \in Z_\omega \setminus Z_1$ and hope to show that they are not in $Z(\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho}))$. If we can show this then we will be done. So in this case we can find some $x_1, x_1' \in \widetilde{\Gamma}$ and distinct $q_1, q_2 \in \mathbb{Q} \setminus \{0\}$ such that

$$x_1 g = q_1 x_1,$$
$$x_1' g = q_2 x_1'.$$

Note that x_1 and x_1' must obviously be distinct. By p.r.s.(1) we can choose some $x_2 \in \widetilde{\Gamma}$ with $\mathrm{v}(x_2) = \mathrm{v}(x_1')$ and $\widetilde{\varrho}(x_2) = \widetilde{\varrho}(x_1)$. By lemma 10.1.3 and the fact that $g \in Z_\omega$ we know that $x_2 g = q_2 x_2$. It also follows from lemma 10.1.3 that $\mathrm{v}(x_1) \neq \mathrm{v}(x_2)$. Because of this and the fact that $\widetilde{\varrho}(x_1) = \widetilde{\varrho}(x_2)$ we can find a residue automorphism $\alpha\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ such that

$$\alpha\colon x_1 \mapsto x_2.$$

But then in this case

$$x_1 \alpha g = x_2 g = q_2 x_2$$

whilst

$$x_1 g \alpha = q_1 x_1 \alpha = q_1 x_2.$$

It follows that $g \notin Z(\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho}))$, and hence that $Z(\mathrm{Aut}(\widetilde{\Gamma}, \widetilde{\varrho})) \subseteq Z_1$ as required. □

Corollary 11.2.4. Suppose Γ is a pseudo-recursively saturated model of Presburger arithmetic with $\mathrm{Res}(\Gamma) \neq \mathbb{Z}$. Then the centre $Z(\mathrm{Aut}(\Gamma)) = \{1\}$.

Proof. This is a direct consequence of theorem 11.2.3 and lemma 11.2.2. □

Lemma 11.2.5. Suppose Γ is countable pseudo-recursively saturated with $\mathrm{Res}(\Gamma) = \mathbb{Z}$. Then G has non-trivial centre.

Proof. Fix some $q \in \mathbb{Q}$. We aim to find some $h \in \widetilde{G}_{\mathrm{v}}$ such that for all $\gamma \in \widetilde{\Gamma}$ we have $\gamma h = q\gamma$.

Let $\gamma_0, \gamma_1, \gamma_2, \ldots$ be an enumeration of the elements of $\widetilde{\Gamma}$. Set $a_0 = \gamma_0$ and $b_0 = q\gamma_0$. Clearly $\widetilde{\varrho}(a_0) = 0 = \widetilde{\varrho}(b_0)$.

Now we construct sets by back-and-forth so that at the n-th stage we have

$$\{a_0, \ldots, a_n\} \quad \text{and} \quad \{b_0, \ldots, b_n\}$$

both strongly independent and so that $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$. Suppose also that $b_i = q a_i$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$. We then consider the next element γ_m in the enumeration for which

$$\gamma_m \notin \langle a_0, \ldots, a_n \rangle .$$

Either γ_m is strongly independent of $\langle a_0, \ldots, a_n \rangle$ or it is not. If it is, then we set $a_{n+1} = \gamma_m$ and $b_{m+1} = q\gamma_m$. Note that in this case we again have $\widetilde{\varrho}(a_{n+1}) = \widetilde{\varrho}(b_{n+1})$. We also have that

$$\mathrm{st}\left(\frac{a_{n+1}}{a_j}\right) = \mathrm{st}\left(\frac{q a_{n+1}}{q a_j}\right) = \mathrm{st}\left(\frac{b_{n+1}}{b_j}\right)$$

for all $j \in \mathbb{N}$ with $1 \leq j \leq n+1$ and in particular, since $\{a_0, \ldots, a_n, a_{n+1}\}$ is strongly independent, we know that $\{b_0, \ldots, b_n, b_{n+1}\}$ is as well.

Now if γ_m is not strongly independent of $\langle a_0, \ldots, a_n \rangle$ then by the Exchange Lemma we can find some a_{n+1} which is, and such that $\gamma_m \in \langle a_0, \ldots, a_n, a_{n+1} \rangle$. We can then use the same technique as above to find a suitable b_{n+1}.

As can be seen in both cases the criteria are satisfied and hence this completes the forth step.

The back step can be seen to be almost identical, with the exception that given b_{n+1} we must choose $a_{n+1} = q^{-1} b_{n+1}$.

We may therefore construct a residue automorphism h in the same way as for theorem 10.1.4 which maps

$$h\colon a_i \mapsto b_i \quad \text{for all } i \in \mathbb{N}.$$

Now for any $\gamma \in \widetilde{\Gamma}$ we know that

$$\gamma = q_1 a_{i_1} + \cdots + q_k a_{i_k}$$

where $k \in \mathbb{N}$ with $q_1, \ldots, q_k \in \mathbb{Q}$ and $i_1, \ldots, i_k \in \mathbb{N}$. But then

$$\begin{aligned} \gamma g &= q_1 a_{i_1} h + \cdots + q_k a_{i_k} h \\ &= q_1 b_{i_1} + \cdots + q_k b_{i_k} \\ &= q(q_1 a_{i_1} + \cdots + q_k a_{i_k}) = q\gamma. \end{aligned}$$

The residue automorphism h is clearly not the identity, yet is central nonetheless. Lifting this to an automorphism $\hat{h}$ of Γ we see that the centre of G is therefore non trivial as required. □

Models for which $\mathrm{Res}(\Gamma) = \mathbb{Z}$ are really the least interesting cases in the context of Presburger arithmetic. For if we look at $\widetilde{\Gamma}$ for such a Γ we see that the residue information is 'lost' or becomes irrelevant, and we are left with a divisible ordered abelian group satisfying the requirements of pseudo-recursive saturation but without additional structure. So although this case will often proffer an exception to our general results—and we will need to be vigilant of the fact—we will nonetheless refrain from dwelling significantly on these exceptions beyond pointing them out.

11.3 Examples

Although in general G has trivial centre, we find that G is not in fact simple, given the various criteria which we have supposed to be applied to Γ. The following proposition implies this, and is shown to be true using two automorphisms established to exist in the previous chapter.

Proposition 11.3.1. If

$$G_{\mathrm{st}} := \left\{ g \in G : \mathrm{st}\left(\frac{\gamma g}{\gamma}\right) = 1 \text{ for all } \gamma \in \Gamma \right\},$$
$$G_{\mathrm{v}} := \left\{ g \in G : \mathrm{v}(\gamma g) = \mathrm{v}(\gamma) \text{ for all } \gamma \in \Gamma \right\},$$

then G_{st} and G_{v} form non-trivial, proper normal subgroups of G with $G_{\mathrm{st}} < G_{\mathrm{v}}$.

Proof. Recall $g_{\mathrm{st}}, g_{\mathrm{v}} \in G$ are automorphisms of Γ, having been constructed previously in lemmas 10.1.5 and 10.1.6 respectively.

Clearly $1 \in G_{\mathrm{st}}$. Suppose $g_1, g_2 \in G_{\mathrm{st}}$ and $\gamma \in \Gamma$. Then

$$\mathrm{st}\left(\frac{\gamma g_1}{\gamma}\right) = 1 \quad \text{and} \quad \mathrm{st}\left(\frac{\gamma g_2}{\gamma}\right) = 1.$$

So

$$\mathrm{st}\left(\frac{\gamma g_1 g_2}{\gamma}\right) = \mathrm{st}\left(\frac{\gamma g_1}{\gamma g_2^{-1}}\right) = \mathrm{st}\left(\frac{\gamma g_1}{\gamma}\right) \cdot \mathrm{st}\left(\frac{\gamma}{\gamma g_2^{-1}}\right) = 1. \tag{11.2}$$

Also

$$\mathrm{st}\left(\frac{\gamma g_1^{-1}}{\gamma}\right) = \mathrm{st}\left(\frac{\gamma}{\gamma g_1}\right) = \mathrm{st}\left(\frac{\gamma g_1}{\gamma}\right)^{-1} = 1. \tag{11.3}$$

Finally, suppose $h \in G$ and let $\gamma' = \gamma h^{-1}$. Then

$$\mathrm{st}\left(\frac{\gamma h^{-1} g_1 h}{\gamma}\right) = \mathrm{st}\left(\frac{\gamma h^{-1} g_1}{\gamma h^{-1}}\right) = \mathrm{st}\left(\frac{\gamma' g_1}{\gamma'}\right) = 1. \tag{11.4}$$

By (11.2), (11.3) and (11.4) above it is clear that $G_{\mathrm{st}} \trianglelefteq G$.

Similarly it is clear that $1 \in G_{\mathrm{v}}$. Suppose $g_1, g_2 \in G_{\mathrm{v}}$ and $\gamma \in \Gamma$. Then

$$\mathrm{v}(\gamma) = \mathrm{v}(\gamma g_1) = \mathrm{v}(\gamma g_1 g_2). \tag{11.5}$$

Also, by setting $\gamma' = \gamma g_1^{-1}$ we have $\mathrm{v}(\gamma') = \mathrm{v}(\gamma' g_1)$ so

$$\mathrm{v}(\gamma) = \mathrm{v}(\gamma g_1^{-1}). \tag{11.6}$$

Finally, suppose $h \in G$ and now set $\gamma' = \gamma h^{-1}$. Then

$$\mathrm{v}(\gamma h^{-1} g_1 h) = \mathrm{v}(\gamma' g_1 h) = \mathrm{v}(\gamma' h) = \mathrm{v}(\gamma h^{-1} h) = \mathrm{v}(\gamma). \tag{11.7}$$

By (11.5), (11.6) and (11.7) above it is clear that $G_{\mathrm{v}} \trianglelefteq G$.

By inspection we see that $G_{\mathrm{st}} \leq G_{\mathrm{v}}$, but $g_{\mathrm{v}} \in G_{\mathrm{v}} \setminus G_{\mathrm{st}}$, so $G_{\mathrm{st}} < G_{\mathrm{v}}$ as required.

Also note that $1 \neq g_{\mathrm{st}} \in G_{\mathrm{st}}$. The result $\{1\} \lhd G_{\mathrm{st}} \lhd G$ then follows from this and the previous result.

Finally, to see that $\{1\} \lhd G_{\mathrm{v}} \lhd G$ we simply need to note that any $g \in G$ which maps an element $\gamma \in \Gamma$ so that $\mathrm{v}(\gamma) \neq \mathrm{v}(\gamma g)$ will not be included in G_{v}. By homogeneity of Γ (cf. corollary 9.3.10) it is immediate that such automorphisms do indeed exist and this completes the proof. □

The subgroups G_{v} and G_{st} are also closed subgroups. This fact will be shown later when we generalise the theorem just proved to cover all of the closed normal subgroups of G.

Also of particular interest is the fact that they are maximum and minimum closed normal subgroups of G respectively. The fact the G_{v} is maximal is made clear by proposition 10.2.3. The result that G_{st} is minimal follows from theorem 11.5.10, which we will now spend some time working towards. In fact this theorem will give us the result which describes all of the closed normal subgroups of G in terms of $\mathrm{stQ}(\Gamma)$.

11.4 stQ-closure

We have seen in previous chapters that several results can be considered in terms of projections of the space $\mathrm{stQ}^{\mathrm{V}}$ onto $(\mathrm{stQ}(\Gamma)_{>0})^n$ for $n \in \omega$. We wish to form a more complete description of the association between an automorphism g and the projections of $\theta(g)$ in terms of n-tuples from $(\mathrm{stQ}(\Gamma)_{>0})^n$ which will be applicable to closed normal subgroups. We do this using the notion of stQ-closure.

Definition 11.4.1. If $S_n \subseteq (\mathrm{stQ}(\Gamma)_{>0})^n \subseteq (\mathbb{R}^*_{>0})^n$ then $S = \bigcup_{n\in\omega} S_n$ is stQ**-closed** if:

1. each S_n is non-empty and closed under pointwise multiplication;

2. each S_n is closed under inversion (where $(r_1, \ldots, r_n)^{-1} = (r_1^{-1}, \ldots, r_n^{-1})$);

3. when $(r_1, \ldots, r_n) \in S$ and $m \leq n$ then $(r_1, \ldots, r_{m-1}, r_{m+1}, \ldots, r_n) \in S$;

4. when $(r_1, \dots, r_n) \in S$ and $m \leq n+1$ then there exists at least one r'_m so that $(r_1, \dots, r_{m-1}, r'_m, r_m, \dots, r_n) \in S$.

It is clear that these conditions can be related to projections applied to some $\theta(g)$. For example, if $v_1, \dots, v_n$ are values of Γ and consider the projection π_n of $\mathrm{stQ}^{\mathrm{V}}$ given by

$$\pi_n \colon \mathrm{stQ}^{\mathrm{V}} \to \left(\mathrm{stQ}_{v_1}, \dots, \mathrm{stQ}_{v_n}\right).$$

Then the map

$$\pi_{n-1} \colon \mathrm{stQ}^{\mathrm{V}} \to \left(\mathrm{stQ}_{v_1}, \dots, \mathrm{stQ}_{v_{m-1}}, \mathrm{stQ}_{v_{m+1}}, \dots, \mathrm{stQ}_{v_n}\right)$$

is also a projection of $\mathrm{stQ}^{\mathrm{V}}$. By applying π_n and π_{n-1} to some $\theta(g)$ this then relates to the third condition, whilst a similar relation can be provided for the fourth. For the first two, we have to consider the case, not of some single automorphism g, but rather the group $\langle g \rangle$ of automorphisms generated from g. We then see that for the projection $\pi_n(\theta(g))$ as applied to $\theta(g)$, applying this to g^{-1} to get $\pi_n(\theta(g^{-1}))$ will relate to the second condition. The first case is a little more complicated, but if it is not already clear it should become so by the end of the chapter.

In order to derive subgroups of G from these stQ-closed sets we use the following definition.

Definition 11.4.2. If $S \subseteq \cup_{n \in \omega} (\mathrm{stQ}(\Gamma)_{>0})^n$ satisfies the stQ-closure properties as given in definition 11.4.1 above, then we define G_S to be the set of residue automorphisms

$$G_S = \left\{ g \in G_{\mathrm{v}} : \forall\, n \in \omega\ \forall\, \mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n) \left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \dots, \mathrm{st}\left(\frac{x_n g}{x_n}\right) \right) \in S \right\}.$$

The notation used suggests that the set S is being 'preserved' in some way by the automorphisms of G_S, yet the automorphisms act on Γ and it is not immediately clear how we might be seen to be preserving S. The answer lies in the description of S as a set of the projections of elements from $\mathrm{stQ}^{\mathrm{V}}$.

Suppose we take any $g \in G_S$. Then $\theta(g)$ is the coloured set of ordered values generated by g. But from the definition of G_S we can see that any projection $\pi_n \colon \mathrm{stQ}^{\mathrm{V}} \to (\mathrm{stQ}(\Gamma)_{>0})^n$ will map $\theta(g)$ to an element of S. Indeed all projections of $\mathrm{stQ}^{\mathrm{V}}$ applied to elements $\theta(g)$ for any $g \in G_S$ will be in S, so by considering the orbit of $\theta(\mathrm{id}_G) = \mathrm{id}_{\mathrm{stQ}^{\mathrm{V}}}$ when we apply the action of G_S to it, we see that all projections of this orbit will be in S.

We conclude that the elements $g \in G_S$ preserve S when they are considered as an action on $\mathrm{stQ}^{\mathrm{V}}$, and S is considered as the image of the projections applied to $\mathrm{stQ}^{\mathrm{V}}$.

In the next few theorems, we intend to show that

1. G_S is a closed normal subgroup of G;

2. if G has trivial centre, then every closed normal subgroup of G is of the form G_S for some stQ-closed $S \subseteq \bigcup_{n\in\omega}(\mathbb{R}^*_{>0})^n$.

Theorem 11.4.3. If $S \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\Gamma)_{>0})^n$ is stQ-closed then G_S is a closed normal subgroup of G.

Proof. We begin by showing that G_S is a subgroup of G. Let $S = \bigcup_{n<\omega} S_n$. Now each S_n is non-empty and closed under inverses, so it is certainly the case that

$$\underbrace{(1, 1, \ldots, 1)}_{n} \in S_n.$$

But for the identity $1 \in G$ we know that

$$\forall\, n < \omega\ \forall\, \mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n) \left(\mathrm{st}\left(\frac{x_1}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{x_n}\right)\right) = (1, \ldots, 1) \in S_n,$$

from which it follows that $1 \in G_S$.

Now suppose $h_1, h_2 \in G_S$. For any $n < \omega$ and $x_1, \ldots, n_n \in \widetilde{\Gamma}$ with $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ we know that

$$\left(\mathrm{st}\left(\frac{x_1 h_1}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n h_1}{x_n}\right)\right) \in S_n \quad \text{and} \quad \left(\mathrm{st}\left(\frac{x_1 h_2}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n h_2}{x_n}\right)\right) \in S_n.$$

But S_n is closed under multiplication, hence

$$\left(\mathrm{st}\left(\frac{x_1 h_1}{x_1}\right) \cdot \mathrm{st}\left(\frac{x_1 h_2}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n h_1}{x_n}\right) \cdot \mathrm{st}\left(\frac{x_n h_2}{x_n}\right)\right) \in S_n$$

and so

$$\left(\mathrm{st}\left(\frac{x_1 h_1 h_2}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n h_1 h_2}{x_n}\right)\right) \in S_n.$$

Since this holds for all $n \in \omega$ and all $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ we therefore have that $h_1 h_2 \in G_S$ as required.

If $h_1 \in G_S$ then for all $n \in \omega$ and $x_1, \ldots, x_n \in \widetilde{\Gamma}$ with $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ we have that

$$\left(\mathrm{st}\left(\frac{x_1 h_1}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n h_1}{x_n}\right)\right) \in S_n.$$

But S_n is closed under inverses. Hence

$$\left(\operatorname{st}\left(\frac{x_1h_1}{x_1}\right)^{-1},\ldots,\operatorname{st}\left(\frac{x_nh_1}{x_n}\right)^{-1}\right)=\left(\operatorname{st}\left(\frac{x_1h_1^{-1}}{x_1}\right),\ldots,\operatorname{st}\left(\frac{x_nh_1^{-1}}{x_n}\right)\right)\in S_n$$

and so $h_1^{-1}\in G_S$ as required.

From the above it follows that $G_S \leq G$.

We now hope to show that G_S is normal in G. So suppose that $h_1 \in G_S$ and $g \in G$. We must show that $g^{-1}h_1g \in G_S$.

Now for any $n < \omega$ and $x_1, \ldots, x_n \in \widetilde{\Gamma}$ with $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ we know that $\mathrm{v}(x_1g^{-1}) < \cdots < \mathrm{v}(x_ng^{-1})$ since g is an automorphism and hence preserves the valuation classes and order. Also $h_1 \in G_S$, and so

$$\left(\operatorname{st}\left(\frac{x_1g^{-1}h_1}{x_1g^{-1}}\right),\ldots,\operatorname{st}\left(\frac{x_ng^{-1}h_1}{x_ng^{-1}}\right)\right)\in S_n.$$

In particular, then,

$$\left(\operatorname{st}\left(\frac{x_1g^{-1}h_1g}{x_1}\right),\ldots,\operatorname{st}\left(\frac{x_ng^{-1}h_1g}{x_n}\right)\right)\in S_n,$$

and so $g^{-1}h_1g \in G_S$ as required.

Finally we must show that G_S is closed in G. So take any $g \in G \setminus G_S$. We want to find a basic open subset $\mathscr{U}$ of G such that $g \in \mathscr{U}$ and $\mathscr{U} \cap G_S = \emptyset$. Now if $g \notin G_S$, then there is some $n < \omega$ and $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ such that

$$\left(\operatorname{st}\left(\frac{x_1g}{x_1}\right),\ldots,\operatorname{st}\left(\frac{x_ng}{x_n}\right)\right)\notin S_n.$$

Let $\bar{x} = (x_1, \ldots, x_n) \in \widetilde{\Gamma}^n$ and let $\mathscr{U}$ be the basic open set

$$\mathscr{U} = \{\, g' \in G : \bar{x}g' = \bar{x}g\}.$$

Clearly $g \in \mathscr{U}$. But suppose $g' \in \mathscr{U}$. Then

$$\left(\operatorname{st}\left(\frac{x_1g'}{x_1}\right),\ldots,\operatorname{st}\left(\frac{x_ng'}{x_n}\right)\right)\notin S_n$$

and hence $g' \notin G_S$. Thus $\mathscr{U} \cap G_S = \emptyset$ as required.

It follows from the above that G_S is a closed normal subgroup of G. □

11.5 All of the normal subgroups

In order to show that every closed normal subgroup of G is of the form G_S for some $S \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\Gamma)_{>0})^n$ we require a fairly complex result concerning conjugates of residue automorphisms which fix initial segments. So as to simplify the presentation of this result we provide some definitions.

Definition 11.5.1. If g is a residue automorphism of $\widetilde{\Gamma}$ we call the set

$$\mathrm{Var}_{\widetilde{\Gamma}}(g) = \{\,\gamma - \gamma g : \gamma \in \widetilde{\Gamma}\}$$

the set of variations by g on $\widetilde{\Gamma}$.

Similarly, if $v \in \mathrm{V}$ we call the set

$$\mathrm{Var}_{<v}(g) = \{\,\gamma - \gamma g : \gamma \in \widetilde{\Gamma}, \mathrm{v}(\gamma) < v\}$$

the set of variations by g of elements with value less than v.

The sets $\mathrm{Var}_{\widetilde{\Gamma}}(g)$ and $\mathrm{Var}_{<\mathrm{v}(s)}(g)$ are both clearly $\mathbb{Q}$-subspaces and hence $\{\,\gamma - \gamma g : \gamma \in \widetilde{\Gamma}\} = \langle \gamma - \gamma g : \gamma \in \widetilde{\Gamma}\rangle$, where the later is the set of linear combinations generated by $\widetilde{\Gamma}$ over $\mathbb{Q}$.

Lemma 11.5.2. Suppose g is a non-trivial residue automorphism and that $s \in \widetilde{\Gamma}$ is an element *not* fixed by g. Then for any $\gamma \in \widetilde{\Gamma}$ with $\mathrm{v}(\gamma) \geq \mathrm{v}(s)$ there exist some $\gamma' \in \widetilde{\Gamma}$ with $\mathrm{v}(\gamma') = \mathrm{v}(\gamma)$ such that $\gamma' g \neq \gamma'$.

Proof. Suppose otherwise. Then for some γ with $\mathrm{v}(\gamma) \geq \mathrm{v}(s)$ we have $\gamma' g = \gamma'$ for all γ' such that $\mathrm{v}(\gamma') = \mathrm{v}(\gamma)$. Since $sg \neq s$ we may suppose $\mathrm{v}(\gamma) > \mathrm{v}(s)$.

Now take any such γ' with $\gamma' g = \gamma'$. Then $\mathrm{v}(\gamma' + s) = \mathrm{v}(\gamma')$ since $\mathrm{v}(s) < \mathrm{v}(\gamma')$. But then

$$(\gamma' + s)g = \gamma' g + sg = \gamma' + sg \neq \gamma' + s,$$

which contradicts our assumption. □

The above lemma 11.5.2 can be re-phrased by saying that on any valuation class which lies above or is equal to that of s the residue automorphism g cannot act like the identity.

Lemma 11.5.3. Suppose that g is a non-trivial residue automorphism which fixes a non-standard initial segment, and that $s \in \widetilde{\Gamma} \setminus \{0\}$. Suppose further that $r \in \mathrm{Res}(\widetilde{\Gamma})$. Then we can find s' with $s' \frown s$ so that $s' - s'g = s - sg$ and $\widetilde{\varrho}(s') = r$. In particular, if $s = sg$ then $s' = s'g$.

Proof. Since the residue r exists in $\widetilde{\Gamma}$ we can find some $\gamma \in \widetilde{\Gamma}$ with $\widetilde{\varrho}(\gamma) = r$. But then $\gamma - s \in \widetilde{\Gamma}$ and so by p.r.s.(1) we can find some element $s'' \in \widetilde{\Gamma}$ so that $\widetilde{\varrho}(s'') = \widetilde{\varrho}(\gamma - s)$ and with $\mathrm{v}(s'') < \mathrm{v}(s)$ so that s'' is contained in the initial segment fixed by g. Let $s' = s + s''$. Then since $\mathrm{v}(s'') < \mathrm{v}(s)$ it is clear that $s' \frown s$. Moreover,

$$\begin{aligned}\widetilde{\varrho}(s') &= \widetilde{\varrho}(s + s'') \\ &= \widetilde{\varrho}(s) + \widetilde{\varrho}(s'') \\ &= \widetilde{\varrho}(s) + \widetilde{\varrho}(\gamma - s) \\ &= \widetilde{\varrho}(\gamma) \\ &= r.\end{aligned}$$

Also,

$$\begin{aligned}s' - s'g &= (s + s'') - (s + s'')g \\ &= s - sg + s'' - s''g \\ &= s - sg\end{aligned}$$

since g fixes s''. The element s' therefore gives the result. □

Although straightforward, the above result will be particularly important. It effectively tells us that under certain conditions the residues have little impact; we are able to construct automorphisms independent from the residues of the model. We will find that it is the remaining structure of the standard parts which plays the central role.

We are now in a position to provide the result mentioned earlier.

Theorem 11.5.4. Let Γ be a countable pseudo-recursively saturated model of Presburger arithmetic. Suppose $h \in \widetilde{G}_{\mathrm{v}}$ is a non-trivial residue automorphism of $\widetilde{\Gamma}$ which fixes some non-standard initial segment and for which $\gamma h \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$. Suppose further that $a_1 < \cdots < a_n \in \widetilde{\Gamma}$ are strongly independent, that $1 \leq k \leq n$ and that $b_k \frown a_k$ is such that $\widetilde{\varrho}(b_k) = \widetilde{\varrho}(a_k)$. Then there exists some $h_k \in \left\langle h^{\widetilde{G}} \right\rangle$ which fixes each a_i with $\mathrm{v}(a_i) \geq \mathrm{v}(a_k)$ except a_k, which maps a_k to b_k and so that $\gamma h_k \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$.

Proof. We consider a basis for $\mathrm{Var}_{\widetilde{\Gamma}}(h)$. Either this basis is finite, or it is infinite. We consider each of the cases separately.

To begin we will consider the infinite case. Choose some element s so that there exist $\gamma_1, \ldots, \gamma_n \in \widetilde{\Gamma}$ with $0 < \mathrm{v}(\gamma_i) < \mathrm{v}(s)$ and $\gamma_1 - \gamma_1 h, \ldots, \gamma_n - \gamma_n h$ linearly independent.

We are able to do this since the basis for $\mathrm{Var}_{\widetilde{\Gamma}}(h)$ is infinite. We aim to find $s_1, \ldots, s_n$ so that

$$s_1 - s_1 h, \ldots, s_n - s_n h$$

are linearly independent, and with $\widetilde{\varrho}(s_i) = \widetilde{\varrho}(a_i)$ and $\mathrm{st}\left(\frac{s_i}{s_j}\right) = \mathrm{st}\left(\frac{a_i}{a_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$.

If $sh \neq s$ let $s'_n = s$, otherwise set $s'_n = s + \gamma_{j_n}$ where $\gamma_{j_n} \in \{\gamma_1, \ldots, \gamma_n\}$ is arbitrarily chosen. It is clear that in either case we have $s'_n h \neq s'_n$. We now apply lemma 11.5.3 to find some $s_n \frown s'_n$ but with the correct residue, *i.e.* so that $\widetilde{\varrho}(s_n) = \widetilde{\varrho}(a_n)$, and with $s_n - s_n h = s'_n - s'_n h$. Set $c_n = s_n - s_n h$. Since $s_n h \frown s_n$ we also know that $\mathrm{v}(c_n) < \mathrm{v}(s_n)$.

Now let s''_{n-1} be any element such that $\mathrm{st}\left(\frac{s_n}{s''_{n-1}}\right) = \mathrm{st}\left(\frac{a_n}{a_{n-1}}\right)$. We know that such elements exist by p.r.s.(2). In the case when $\mathrm{v}(a_{n-1}) < \mathrm{v}(a_n)$ we must also ensure that $\mathrm{v}(c_n) < \mathrm{v}(s''_{n-1}) < \mathrm{v}(s_n)$, but this can clearly be done by p.r.s.(3). We wish to find some s'_{n-1} so that $s'_{n-1} \frown s''_{n-1}$ and with $s'_{n-1} - s'_{n-1}h$ linearly independent from $s_n - s_n h$. Set $c'_{n-1} = s''_{n-1} - s''_{n-1}h$. If c'_{n-1} is already linearly independent then we simply set $s'_{n-1} = s''_{n-1}$ and are done. So suppose otherwise. Then $c'_{n-1} = q_n c_n$ for some $q_n \in \mathbb{Q} \setminus \{0\}$. Since $\gamma_1 - \gamma_1 h, \ldots, \gamma_n - \gamma_n h$ are linearly independent there can be at most one of these γ_j for which $\gamma_j \in \langle c_n \rangle$. In particular, then, it is straightforward to find one which is not; $\gamma_{j_{n-1}}$, say. We then set $s'_{n-1} = s''_{n-1} + \gamma_{j_{n-1}}$ so that

$$\begin{aligned} s'_{n-1} - s'_{n-1}h &= (s''_{n-1} - s''_{n-1}h) + (\gamma_{j_{n-1}} - \gamma_{j_{n-1}}h) \\ &= q_n c_n + (\gamma_{j_{n-1}} - \gamma_{j_{n-1}}h) \\ &\notin \langle c_n \rangle . \end{aligned}$$

Set $c_{n-1} = s'_{n-1} - s'_{n-1}h$. Again we apply lemma 11.5.3 in order to find some $s_{n-1} \frown s'_{n-1}$ but with residue $\widetilde{\varrho}(s_{n-1}) = \widetilde{\varrho}(a_{n-1})$. Note that by this lemma it is also the case that $s_{n-1} - s_{n-1}h = c_{n-1}$.

We then continue with this process iteratively as follows. Suppose at the i-th stage we have $c_n, \ldots, c_{n-i+1}$ all linearly independent. We wish to find s_{n-i} of the required sort so that $s_{n-i} - s_{n-i}h$ is linearly independent of the $c_n, \ldots, c_{n-i+1}$. So take s''_{n-i} to be any element with $\mathrm{st}\left(\frac{s_{n-i+1}}{s''_{n-i}}\right) = \mathrm{st}\left(\frac{a_{n-i+1}}{a_{n-i}}\right)$. If $\mathrm{v}(a_{n-i}) < \mathrm{v}(a_{n-i+1})$ then we must ensure that $\max\{\mathrm{v}(c_j) : n \geq j > n-i\} < \mathrm{v}(s''_{n-i}) < \mathrm{v}(s_{n-i+1})$, which we can do by p.r.s.(3). If $s''_{n-i} - s''_{n-i}h$ is linearly independent of $c_n, \ldots, c_{n-i+1}$ then we set $s'_{n-i} = s''_{n-i}$ and are done. Otherwise we note that

$$\gamma_1 - \gamma_1 h, \ldots, \gamma_n - \gamma_n h$$

are linearly independent and that $n > i-1$. Hence we are guaranteed that there will be some element $\gamma_{j_{n-i}} \in \{\gamma_1, \dots, \gamma_n\}$ so that $\gamma_{j_{n-i}} - \gamma_{j_{n-i}} h$ is linearly independent of $c_n, \dots, c_{n-i+1}$. We can then set $s'_{n-i} = s''_{n-i} + \gamma_{j_{n-i}}$ and $c_{n-i} = s'_{n-i} - s'_{n-i}h$. Finally apply lemma 11.5.3 to find $s_{n-i} \frown s'_{n-i}$ with $\widetilde{\varrho}(s_{n-i}) = \widetilde{\varrho}(a_{n-i})$. Again by the lemma we know that $s_{n-i} - s_{n-i}h = c_{n-i}$ and we are done.

Continuing this iterative process we will eventually have strongly independent elements $s_1, \dots, s_n$ so that if $c_i = s_i - s_i h$ for all $i \in \mathbb{N}$ with $1 \le i \le n$, then $c_1, \dots, c_n$ are linearly independent. By our construction we also have that $\mathrm{v}(c_i) < \mathrm{v}(s_j)$ for all $i, j \in \mathbb{N}$ with $1 \le i, j \le n$. Now starting with c_1 we can apply the exchange lemma iteratively to $c_1, \dots, c_n$ in order to find strongly independent elements $d_1, \dots, d_n$ and so that

$$s_i - s_i h \in \langle d_1, \dots, d_i \rangle$$

for each $i \in \mathbb{N}$ with $1 \le i \le n$. The fact that $c_1, \dots, c_n$ are linearly independent means that by iterating this process sequentially we will also ensure that

$$s_i - s_i h \notin \langle d_1, \dots, d_{i-1} \rangle \tag{11.8}$$

for $i \in \mathbb{N}$ with $1 \le i \le n$. From the same result for the c_i's we see that $\mathrm{v}(d_i) < \mathrm{v}(s_j)$ for $i, j \in \mathbb{N}$ with $1 \le i, j \le n$. We must check a couple of things. Suppose at some stage of this iterative process $s_i - s_i h = q_1 d_1 + \dots + q_i d_i$. First we wish to show that $s_i h \notin \langle s_1, \dots, s_i, d_1, \dots, d_{i-1} \rangle$. For a contradiction let

$$s_i h = q'_1 s_1 + \dots + q'_i s_i + q''_1 d_1 + \dots + q''_{i-1} d_{i-1}.$$

Adding $s_i - s_i h$ to this we get

$$s_i = q'_1 s_1 + \dots + q'_i s_i + (q_1 + q''_1) d_1 + \dots + (q_{i-1} + q''_{i-1}) d_{i-1} + q_i d_i.$$

But all of these elements are strongly independent, from which it clearly follows that $q_i = 0$, contradicting (11.8).

Second we wish to show that $s_i h \notin \langle s_1, \dots, s_{i-1}, d_1, \dots, d_i \rangle$. Again, for a contradiction let

$$s_i h = q'_1 s_1 + \dots + q'_{i-1} s_{i-1} + q''_1 d_1 + \dots + q''_i d_i.$$

Adding $s_i - s_i h$ we get

$$q'_1 s_1 + \dots + q'_{i-1} s_{i-1} - s_i + (q_1 + q''_1) d_1 + \dots + (q_i + q''_i) d_i = 0,$$

which contradicts the strong independence of $s_1, \dots, s_i, d_1, \dots, d_i$.

We therefore have that

$$s_i h \notin \langle s_1, \dots, s_i, d_1, \dots, d_{i-1} \rangle \cup \langle s_1, \dots, s_{i-1}, d_1, \dots, d_i \rangle$$

and so can use proposition 10.4.5 to produce a word

$$h_k = g_k^{-1} h g_k {g'_k}^{-1} h^{-1} g'_k$$

of the required sort. Since $\gamma h \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$ and h_k consists of a product of conjugates of h it is clear that $\gamma h_k \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$ as required.

We must now consider the finite case. We aim to find some $s_i \in \widetilde{\Gamma}$ with $\widetilde{\varrho}(a_i) = \widetilde{\varrho}(s_i)$ and $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{s_i}{s_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$. We intend to find them so that $s_i h = s_i$ for all i with $\mathrm{v}(a_i) \geq \mathrm{v}(a_k)$ except a_k, and $s_k h \neq s_k$.

We know that the basis for $\mathrm{Var}_{\widetilde{\Gamma}}(h)$ is finite. It is therefore possible to find some element $s \in \widetilde{\Gamma}$ for which

$$\mathrm{Var}_{<\mathrm{v}(s)}(h) = \mathrm{Var}_{\widetilde{\Gamma}}(h).$$

By lemma 11.5.2 we may assume that $s \neq sh$. We can therefore use lemma 11.5.3 to find some $s_k \frown s$ with $\widetilde{\varrho}(s_k) = \widetilde{\varrho}(a_k)$ and $s_k \neq s_k h$.

We now consider the elements a_i where $1 \leq i \leq n$ and for which $\mathrm{v}(a_i) = \mathrm{v}(a_k)$. Let $s''_i \in \widetilde{\Gamma}$ be such that $\mathrm{st}\left(\frac{s_k}{s''_i}\right) = \mathrm{st}\left(\frac{a_k}{a_i}\right)$. Clearly $s''_i - s''_i h \in \mathrm{Var}_{\widetilde{\Gamma}}(h) = \mathrm{Var}_{<\mathrm{v}(s)}(h)$, hence we can find some γ_i with $\mathrm{v}(\gamma_i) < \mathrm{v}(s''_i)$ and $\gamma_i - \gamma_i h = s''_i - s''_i h$. Let $s'_i = s''_i - \gamma_i$. We then have

$$\begin{aligned} s'_i - s'_i h &= (s''_i - s''_i h) - (\gamma_i - \gamma_i h) \\ &= 0. \end{aligned}$$

In other words, we have that $s'_i = s'_i h$. We also note that since $\mathrm{v}(\gamma_i) < \mathrm{v}(s''_i)$ it must be the case that $s'_i \frown s''_i$. Again we then use lemma 11.5.3 to find $s_i \frown s'_i$ with $\widetilde{\varrho}(s_i) = \widetilde{\varrho}(a_i)$. By this lemma we know that $s_i = s_i h$ as required.

The above construction will also work for all a_i with $1 \leq i \leq n$ and $\mathrm{v}(a_i) > \mathrm{v}(a_k)$. For each such a_i we can therefore find some s_i of the required sort, so that $s_i = s_i h$ with $\widetilde{\varrho}(s_i) = \widetilde{\varrho}(a_i)$ and $\mathrm{st}\left(\frac{s_j}{s_i}\right) = \mathrm{st}\left(\frac{a_j}{a_i}\right)$ for all $j \in \mathbb{N}$ with $1 \leq j \leq n$.

Now we intend to map the element $s_k - s_k h$ to the element $a_k - b_k$ (Note that if the signs of $s_k - s_k h$ and $a_k - b_k$ do not agree, we must swap all occurrences of a_k and b_k in the succeeding construction). Since $s_k \frown s_k h$ and $a_k \frown b_k$ we know that

$\mathrm{v}(s_k - s_k h) < \mathrm{v}(s_k)$ and $\mathrm{v}(a_k - b_k) < \mathrm{v}(a_k)$. We also know that $s_k - s_k h \neq 0$ and that $a_k - b_k \neq 0$. Hence it is clear that using theorem 10.1.1 we can find a residue automorphism g_k which maps:

$$g_k\colon \begin{array}{ccll} s_i & \mapsto & a_i & \text{where } 1 \leq i \leq n,\ i \neq k \text{ and } \mathrm{v}(a_i) \leq \mathrm{v}(a_k)\ ; \\ s_k & \mapsto & a_k & ; \\ s_k - s_k h & \mapsto & a_k - b_k & . \end{array}$$

We now set $h_k = g_k^{-1} h g_k$ and claim that this residue automorphism will satisfy our requirements. To see this we first consider the a_i with $\mathrm{v}(a_i) \geq \mathrm{v}(a_k)$ and $a_i \neq a_k$. In these cases we have

$$\begin{aligned} a_i h_k &= a_i g_k^{-1} h g_k \\ &= s_i h g_k, \end{aligned}$$

but by the construction we know that $s_i h = s_i$, so

$$\begin{aligned} a_i h_k &= s_i g_k \\ &= a_i \end{aligned}$$

as required. For the element a_k we have

$$\begin{aligned} a_k h_k &= a_k g_k^{-1} h g_k \\ &= s_k h g_k \\ &= (s_k g_k) - (a_k - b_k) \\ &= b_k \end{aligned}$$

which again is what we require. For the remaining elements we again note, due to h_k being a conjugate of h, that $\gamma h_k \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$. The map h_k therefore satisfies all of our requirements. $\square$

Corollary 11.5.5. Let Γ be a countable pseudo-recursively saturated model of Presburger arithmetic. Suppose h is a non-trivial residue automorphism of $\widetilde{\Gamma}$ which fixes some non-standard initial segment and for which $\gamma h \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$. Suppose further that we have two strongly independent subsets of $\widetilde{\Gamma}$

$$\{a_1, \ldots, a_n\} \quad \text{and} \quad \{b_1, \ldots, b_n\}$$

so that $a_i \frown b_i$ with $\widetilde{\varrho}(a_i) = \widetilde{\varrho}(b_i)$ and $\operatorname{st}\left(\frac{a_i}{a_j}\right) = \operatorname{st}\left(\frac{b_i}{b_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$. Then there exists some $w \in \left\langle h^{\widetilde{G}} \right\rangle$ which maps $w\colon a_i \mapsto b_i$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$.

Proof. We prove this inductively using theorem 11.5.5. Suppose without loss of generality that $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$. For the base case we use theorem 11.5.5 to find a residue automorphism h_n which maps a_n to b_n and such that $a_i \frown a_i h_n$ for all $i \in \mathbb{N}$ with $1 \leq i < n$.

At the i-th step we suppose we have residue automorphisms $h_n, \ldots, h_{n-i+1}$ so that if $w_i = h_n \cdots h_{n-i+1}$ then $a_j w_i = b_j$ for $n - i < j \leq n$ and $a_j w_i \frown a_j$ for $j \in \mathbb{N}$ with $1 \leq j \leq n - i$. We then use theorem 11.5.5 to find some h_{n-i} which maps a_{n-i} to b_{n-i}, fixes all b_j with $j > n - i$ so that $b_j h_{n-i} = b_j$ and so that $(a_j w_i) h_{n-1} \frown (a_j w_i)$ for all $j \in \mathbb{N}$ with $1 \leq j < n - i$. Setting $w_{i+1} = w_i h_{n-i}$ we therefore see that $a_j w_{i+1} = b_j$ for all $j \in \mathbb{N}$ with $n - i \leq j \leq n$ and that $a_j w_{i+1} \frown a_j$ for $j \in \mathbb{N}$ such that $1 \leq j < n - i$. This therefore completes the inductive step.

Continuing the induction up to $i = n$ we ultimately set $w = w_n = h_n \cdots h_1$ to find the residue automorphism required. □

Lemma 11.5.6. Let Γ be a countable pseudo-recursively saturated model of Presburger arithmetic. Suppose $\widetilde{G}$ has trivial centre, $\widetilde{H} \trianglelefteq \widetilde{G}$ and $\widetilde{H} \leq \widetilde{G_{\mathrm{v}}}$. If there is some non-trivial element of $\widetilde{H}$ fixing a non-standard initial segment of $\widetilde{\Gamma}$, then there exists a non-trivial element $h \in \widetilde{H}$ such that $\gamma h \frown \gamma$ for all $\gamma \in \widetilde{\Gamma}$ which also fixes a non-standard initial segment.

Proof. Take any non-trivial element $g_1 \in \widetilde{H}$. Since $\widetilde{G}$ has trivial centre, we can find some $g_2 \in \widetilde{G}$ which does not commute with g_1, *i.e.* so that

$$g_1 g_2 \neq g_2 g_1.$$

Referring to theorem 11.2.3 we see that we are able to assume that g_2 preserves values. We then claim that setting

$$h = 1^{-1} g_1 1 \cdot g_2^{-1} g_1^{-1} g_2 = g_1 g_2^{-1} g_1^{-1} g_2$$

will suffice. To check this we first note that if $\gamma \in \widetilde{\Gamma}$ then

$$\begin{aligned}
\operatorname{st}\left(\frac{\gamma h}{\gamma}\right) &= \operatorname{st}\left(\frac{\gamma g_1 g_2^{-1} g_1^{-1} g_2}{\gamma}\right) \\
&= \operatorname{st}\left(\frac{\gamma g_1}{\gamma}\right) \cdot \operatorname{st}\left(\frac{\gamma}{\gamma g_2}\right) \cdot \operatorname{st}\left(\frac{\gamma g_1^{-1}}{\gamma}\right) \cdot \operatorname{st}\left(\frac{\gamma}{\gamma g_2^{-1}}\right) \\
&= 1.
\end{aligned}$$

We must also ensure that $h \neq 1$. We know that $g_1 g_2 \neq g_2 g_1$ and so $g_1 g_2^{-1} \neq g_2^{-1} g_1$. But if $h = 1$ then $g_1 g_2^{-1} g_1^{-1} g_2 = 1$ so $g_1 g_2^{-1} = g_2^{-1} g_1$, contradicting this. We must therefore have $h \neq 1$ as required.

Finally we note that h will also fix an initial segment of $\widetilde{\Gamma}$. We can see this by setting $s = \min\{s', s' g_2, s' g_2^{-1}\}$, where s' is contained in the initial segment fixed by g_1. Since $s < s'$ it is clear that s is also fixed by g_1. But then for any element γ with $0 < \gamma < s$ we have

$$\gamma h = \gamma g_1 g_2^{-1} g_1^{-1} g_2 = \gamma g_2^{-1} g_1^{-1} g_2.$$

Now $\gamma < s$ so $\gamma g_2^{-1} < s g_2^{-1}$ and $s \leq s' g_2$ so $s g_2^{-1} \leq s' g_2 g_2^{-1} = s'$. Hence $\gamma g_2^{-1} < s'$ and so is fixed by g_1^{-1}. Thus

$$\gamma h = (\gamma g_2^{-1}) g_1^{-1} g_2 = \gamma g_2^{-1} g_2 = \gamma.$$

Since $s > 0$ it follows therefore that h fixes a non-standard initial segment of $\widetilde{\Gamma}$. □

This completes the results which we need in order to prove our final claim that for G with trivial centre, every closed normal subgroup of G is of the form G_S. The next few results will show this.

Theorem 11.5.7. Suppose $N \trianglelefteq G$ is a normal subgroup of residue automorphisms. Let

$$S = \left\{ \left(\operatorname{st}\left(\frac{x_1 h}{x_1}\right), \ldots, \operatorname{st}\left(\frac{x_n h}{x_n}\right) \right) : n \in \omega, h \in N, \operatorname{v}(x_1) < \cdots < \operatorname{v}(x_n) \right\}.$$

Then S satisfies the stQ-closure properties and $N \subseteq G_S$.

Proof. We take each of the claims in turn. To begin we check each of the four stQ-closure properties.

1. If $(r_1, \dots, r_n) \in S$ and $(r'_1, \dots, r'_n) \in S$, then there are residue automorphisms $h_1, h_2 \in N$ so that for some $x_1, \dots, x_n$ and $x'_1, \dots, x'_n$ with $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ and $\mathrm{v}(x'_1) < \cdots < \mathrm{v}(x'_n)$ we have

$$(r_1, \dots, r_n) = \left(\mathrm{st}\left(\frac{x_1 h_1}{x_1}\right), \dots, \mathrm{st}\left(\frac{x_n h_1}{x_n}\right)\right);$$
$$(r'_1, \dots, r'_n) = \left(\mathrm{st}\left(\frac{x'_1 h_2}{x'_1}\right), \dots, \mathrm{st}\left(\frac{x'_n h_2}{x'_n}\right)\right).$$

Choose $y'_1, \dots, y'_n$ so that $\mathrm{v}(y'_i) = \mathrm{v}(x'_i)$ and $\widetilde{\varrho}(y'_i) = \widetilde{\varrho}(x_i)$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$. It is possible to do this by p.r.s.(1). By lemma 10.1.3 we know that

$$(r'_1, \dots, r'_n) = \left(\mathrm{st}\left(\frac{y'_1 h_2}{y'_1}\right), \dots, \mathrm{st}\left(\frac{y'_n h_2}{y'_n}\right)\right).$$

By theorem 10.1.1 we can find a residue automorphism $g\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ which maps $g\colon y'_i \mapsto x_i$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$.

Now N is a normal subgroup, so $g^{-1}h_2 g h_1 \in N$. But for $i \in \mathbb{N}$ with $1 \leq i \leq n$ we have

$$\begin{aligned}
\mathrm{st}\left(\frac{x_i g^{-1} h_2 g h_1}{x_i}\right) &= \mathrm{st}\left(\frac{x_i g^{-1} h_2 g}{x_i}\right) \cdot \mathrm{st}\left(\frac{x_i h_1}{x_i}\right) \\
&= \mathrm{st}\left(\frac{x_i g^{-1} h_2}{x_i g^{-1}}\right) \cdot \mathrm{st}\left(\frac{x_i h_1}{x_i}\right) \\
&= \mathrm{st}\left(\frac{y'_i h_2}{y'_i}\right) \cdot \mathrm{st}\left(\frac{x_i h_1}{x_i}\right) \\
&= r'_i \cdot r_i.
\end{aligned}$$

Hence $(r_1 \cdot r'_1, \dots, r_n \cdot r'_n) \in S$ as required. This last part can also be proven using proposition 10.4.4.

2. Let $h_1 \in N$ be as before. Then $h_1^{-1} \in N$ and

$$\begin{aligned}
\left(\mathrm{st}\left(\frac{x_1 h_1^{-1}}{x_1}\right), \dots, \mathrm{st}\left(\frac{x_n h_1^{-1}}{x_n}\right)\right) &= \left(\mathrm{st}\left(\frac{x_1}{x_1 h_1}\right), \dots, \mathrm{st}\left(\frac{x_n}{x_n h_1}\right)\right) \\
&= (r_1^{-1}, \dots, r_n^{-1}) \\
&\in S.
\end{aligned}$$

3. Suppose $(r_1, \dots, r_n) \in S$ and $h_1 \in N$ are as before. If $k \in \mathbb{N}$ is such that $1 \leq k \leq n$, then $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_{k-1}) < \mathrm{v}(x_{k+1}) < \cdots < \mathrm{v}(x_n)$ so

$$\left(\mathrm{st}\left(\frac{x_1 h_1}{x_1}\right), \dots, \mathrm{st}\left(\frac{x_{k-1} h_1}{x_{k-1}}\right), \mathrm{st}\left(\frac{x_{k+1} h_1}{x_{k+1}}\right), \dots, \mathrm{st}\left(\frac{x_n h_1}{x_n}\right)\right) \in S.$$

So $(r_1, \dots, r_{k-1}, r_{k+1}, \dots, r_n) \in S$ as required.

4. Suppose $(r_1, \ldots, r_n) \in S$ and $h_1 \in N$ are as before. If $k \in \mathbb{N}$ is such that $1 \leq k \leq n+1$ then by p.r.s.(3) we can find some $x \in \widetilde{\Gamma}$ so that $\mathrm{v}(x_{k-1}) < \mathrm{v}(x) < \mathrm{v}(x_k)$. Let $r = \mathrm{st}\left(\frac{xh_1}{x}\right)$. Then

$$\left(\mathrm{st}\left(\frac{x_1h_1}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_{k-1}h_1}{x_{k-1}}\right),\ \mathrm{st}\left(\frac{xh_1}{x}\right), \mathrm{st}\left(\frac{x_kh_1}{x_k}\right), \ldots, \mathrm{st}\left(\frac{x_nh_1}{x_n}\right)\right)$$
$$= (r_1, \ldots, r_{k-1}, r, r_k, \ldots, r_n) \in S$$

as required.

So S satisfies the stQ-closure properties. We must now show that $N \subseteq G_S$.

So suppose $h \in N$. We wish to show that $h \in G_S$. But this is clear by the construction of S, as for any $n \in \omega$ and $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ it is clearly the case that

$$\left(\mathrm{st}\left(\frac{x_1h}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_nh}{x_n}\right)\right) \in S$$

Hence by the definition of G_S (definition 11.4.2) we see that $h \in G_S$. □

Theorem 11.5.8. Suppose $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are strongly independent with $\widetilde{\varrho}(a_i) = \widetilde{\varrho}(b_i)$ and $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$ for all $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$. Then there exists a residue automorphism g which maps a_i to b_i for all $i \in \mathbb{N}$ with $1 \leq i \leq n$ and fixes some non-standard initial segment.

Proof. Begin by choosing some element γ with $0 < \mathrm{v}(\gamma) < \mathrm{v}(a_i), \mathrm{v}(b_i)$ for all $i \in \mathbb{N}$ and $1 \leq i \leq n$, which can be done by p.r.s.(3). Having done this it is then possible to construct a residue automorphism fixing all elements with value less than γ using a standard back-and-forth construction, as in theorem 10.1.4 and its corollary 10.1.5. □

Lemma 11.5.9. Suppose G has trivial centre and $1 \neq N \trianglelefteq G$. Then N contains a non-trivial element which fixes a non-standard initial segment.

Proof. Take any non-trivial element $h \in \widetilde{N}$. Since it is non-trivial we know that $a_1h = a_2$ for some $a_1, a_2 \in \widetilde{\Gamma}$ with $a_1 \neq a_2$. Since G has trivial centre, there exist elements in Γ with non-zero residue and we can find such an element between a_1 and a_2. For such an element it is clear that it cannot be mapped to itself and hence we may assume without loss of generality that $\widetilde{\varrho}(a_1) \neq 0$. It follows from this that a_1 and a_2 are linearly independent. So by the exchange lemma we can find some a_3 so that $\{a_2, a_3\}$ is strongly independent and $a_1 \in \langle a_2, a_3 \rangle$. Now choose some $a_4 \neq a_3$ with $a_4 \frown a_3$ and $\widetilde{\varrho}(a_4) = \widetilde{\varrho}(a_3)$. Such an element exists by p.r.s.(1). Then $\{a_2, a_4\}$ is also strongly independent and so by theorem 11.5.8 we can construct a residue automorphism g which fixes a non-standard initial segment and so that

$$g\colon \begin{array}{ccc} a_2 & \mapsto & a_2 \\ a_3 & \mapsto & a_4 \end{array} \begin{array}{c} ; \\ . \end{array}$$

We then set $h' = g^{-1}h^{-1}gh$. Clearly $h' \in \widetilde{N}$ since $\widetilde{N}$ is normal in $\widetilde{G}$ and we also note that h' must fix some non-standard initial segment, since g does. If we can show that h' is not the identity we will therefore be done.

Now since $a_1 \in \langle a_2, a_3 \rangle$ we can write $a_1 = q_2a_2 + q_3a_3$ or some $q_2, q_3 \in \mathbb{Q}$ where $q_3 \neq 0$. But then

$$\begin{aligned} a_2h' &= a_2g^{-1}h^{-1}gh \\ &= a_1gh \\ &= (q_2a_2 + q_3a_3)gh \\ &= (q_2a_2 + q_3a_4)h \\ &\neq a_1h = a_2. \end{aligned}$$

So since $a_2h' \neq a_2$ we know that h' is not the identity, which completes the proof. □

Theorem 11.5.10. Suppose that G has trivial centre and that N is a closed normal subgroup of G. Let

$$S = \left\{ \left(\operatorname{st}\left(\frac{x_1h}{x_1}\right), \dots, \operatorname{st}\left(\frac{x_nh}{x_n}\right) \right) : n \in \omega, h \in N, \mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n) \right\}.$$

Then $N = G_S$.

Proof. We know by the previous theorem 11.5.7 that $N \subseteq G_S$. We must show that $G_S \subseteq N$. We will suppose otherwise and aim to show that in this case, N is not closed. So suppose $N \subset G_S$. Let $g \in G_S \setminus N$. We will show that every basic open set containing g also contains an element of N.

So let $\bar{x} \in \Gamma$, then the basic open set $G_{\bar{x}}g$ is given by

$$G_{\bar{x}}g = \{ \alpha \in G : \bar{x}\alpha = \bar{x}g \}.$$

Let $\bar{x} = (x_1, \dots, x_n)$. We hope to find an element $h_1 \in N$ so that $x_ih_1 \frown x_ig$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$. To do this we may assume without loss of generality that

$x_1 < \cdots < x_n$, however we do not necessarily know that $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$. We therefore take a representative, x_{l_i}, from each valuation class so that

$$\mathrm{v}(x_{l_1}) < \mathrm{v}(x_{l_2}) < \cdots < \mathrm{v}(x_{l_m})$$

for some $l_1 < l_2 < \cdots < l_m$ and

$$\{\mathrm{v}(x_{l_1}), \ldots, \mathrm{v}(x_{l_m})\} = \{\mathrm{v}(x_1), \ldots, \mathrm{v}(x_n)\}.$$

Then since $g \in G_S$ it must be the case that

$$\left(\mathrm{st}\left(\frac{x_{l_1}g}{x_{l_1}}\right), \ldots, \mathrm{st}\left(\frac{x_{l_m}g}{x_{l_m}}\right)\right) \in S.$$

We set

$$\left(\mathrm{st}\left(\frac{x_{l_1}g}{x_{l_1}}\right), \ldots, \mathrm{st}\left(\frac{x_{l_m}g}{x_{l_m}}\right)\right) = (r_1, \ldots, r_m).$$

Hence there must be some element $h_3 \in N$ and $y_1, \ldots, y_m$ with $\mathrm{v}(y_1) < \cdots < \mathrm{v}(y_m)$ so that

$$\left(\mathrm{st}\left(\frac{y_1 h_3}{y_1}\right), \ldots, \mathrm{st}\left(\frac{y_m h_3}{y_m}\right)\right) = (r_1, \ldots, r_m).$$

Then by proposition 10.4.4 we can find some residue automorphism $h_1 \in \left\langle {h_3}^G \right\rangle \subseteq N$ so that $x_{l_i}h_1 \frown x_{l_i}g$ for $i \in \mathbb{N}$ with $1 \leq i \leq m$. But then by lemma 10.1.3 we must have $x_i h_1 \frown x_i g$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$.

Now by lemma 11.5.9 we know that N must contain a non-trivial element which fixes a non-standard initial segment.

So by lemma 11.5.6, since G has trivial centre, we can find a non-trivial $h_4 \in N$ which fixes a non-standard initial segment and such that $\gamma h_4 \frown \gamma$ for all $\gamma \in \Gamma$. Then by corollary 11.5.5 we can find some $h_2 \in \left\langle {h_4}^G \right\rangle \subseteq N$ so that $h_2 \colon x_i h_1 \mapsto x_i g$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$. We set $h = h_1 h_2 \in N$ so that $\bar{x}h = \bar{x}g$ and so clearly $h \in N \cap G_{\bar{x}}g$ which gives us the result.

We therefore have $N \cap G_{\bar{x}}g \neq \emptyset$, from which we may conclude that N is not closed, contradicting our assumption. If N is a closed normal subgroup of G, we must therefore have that $N = G_S$. □

Chapter 12

The Lattice of Normal Subgroups

12.1 Orbits

The results in the previous chapter concerning normal subgroups can be written in the language of orbits by considering the residue automorphisms to be group actions on $\widetilde{\Gamma}$. We do this by defining an equivalence class as follows:

Definition 12.1.1. Suppose $N \trianglelefteq G$ is closed. Then for any $\bar{x}, \bar{y} \in \Gamma$ we write

$$\bar{x} \sim_N \bar{y}$$

if there exists some $h \in N$ for which $\bar{x}h = \bar{y}$.

Note that the equivalence classes produced by $\sim_G$ constitute precisely those classes produced by types, so that

$$\bar{x} \sim_G \bar{y} \quad \text{if and only if} \quad \mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})\,.$$

Alternatively these can be seen to be the parameter-free $\mathscr{L}_{\infty,\omega}$-definable equivalence relations on Γ^n for $n \in \mathbb{N}$. This is a general relationship between the closed normal subgroups of the topology and equivalence relations, not solely restricted to the specific case of Presburger arithmetic which we are interested in.

The main result can then be given as follows:

Proposition 12.1.2. If $N = G_S$ and $\bar{x}, \bar{y} \in \Gamma$, then

$$\bar{x} \sim_N \bar{y}$$

if and only if

$$\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y}) \quad \text{and} \quad \left(\mathrm{st}\left(\frac{y_{i_1}}{x_{i_1}}\right), \ldots, \mathrm{st}\left(\frac{y_{i_m}}{x_{i_m}}\right)\right) \in S$$

where $m \leq n$ with $i_1, \ldots, i_m \in \{1, \ldots, n\}$ so that $\mathrm{v}(y_{i_1}) < \cdots < \mathrm{v}(y_{i_m})$ and

$$\{\mathrm{v}(y_{i_1}), \ldots, \mathrm{v}(y_{i_m})\} = \{\mathrm{v}(y_1), \ldots, \mathrm{v}(y_n)\}.$$

Proof. This is clear from the definition of G_S. □

12.2 Ordering and inclusion relations

Proposition 12.2.1. Suppose Γ is a countable pseudo-recursively saturated model of Presburger arithmetic and that $T_1 \subseteq \bigcup_{n \in \omega} (\mathrm{stQ}(\Gamma)_{>0})^n$ is stQ-closed. Suppose further that $\bar{x}, \bar{y} \in \Gamma^n$ are such that $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ with $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ and

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) = (r_1, \ldots, r_n) \in T_1.$$

Then there exists some automorphism $g \colon \Gamma \to \Gamma$ such that $g \colon x_i \mapsto y_i$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$, and $g \in G_{T_1}$.

Proof. Let $\gamma_1, \gamma_2, \ldots$ be an enumeration of $\widetilde{\Gamma}$. We suppose without loss of generality that for the first n elements of the enumeration $\gamma_i = x_i$ and set $\gamma_i' = y_i$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$. Clearly

$$0 < \mathrm{v}(\gamma_1) < \mathrm{v}(\gamma_2) < \cdots < \mathrm{v}(\gamma_n)$$

and $\mathrm{tp}(\gamma_1, \ldots, \gamma_n) = \mathrm{tp}(\gamma_1', \ldots, \gamma_n')$. We note that

$$\{\gamma_1, \ldots, \gamma_n\} \quad \text{and} \quad \{\gamma_1', \ldots, \gamma_n'\}$$

are strongly independent sets.

We now continue the construction using back-and-forth. So, suppose we have sets

$$\{\gamma_1, \ldots, \gamma_m\} \quad \text{and} \quad \{\gamma_1', \ldots, \gamma_m'\}$$

both strongly independent with $m \geq n$ and so that for all $\mathrm{v}(\gamma_{i_1}) < \cdots < \mathrm{v}(\gamma_{i_s})$ there is $(r_{i_1}, \ldots, r_{i_s}) \in S_1$ where $\mathrm{st}\left(\frac{\gamma_{i_j}}{\gamma_{i_j}'}\right) = r_{i_j}$. We also suppose that $\widetilde{\varrho}(\gamma_i) = \widetilde{\varrho}(\gamma_i')$ and $\mathrm{st}\left(\frac{\gamma_i}{\gamma_j}\right) = \mathrm{st}\left(\frac{\gamma_i'}{\gamma_j'}\right)$ for $i, j \in \mathbb{N}$ with $1 \leq i, j \leq m$.

For the back-and-forth step, take the next γ_t not yet included in

$$\langle \gamma_1, \ldots, \gamma_m \rangle .$$

There are three possible situations which may occur:

1. γ_t is linearly independent of $\{\gamma_1, \dots, \gamma_m\}$ but is not strongly independent;

2. γ_t is strongly independent of $\{\gamma_1, \dots, \gamma_m\}$ but $\mathrm{v}(\gamma_t) = \mathrm{v}(\gamma_j)$ for some $j \in \mathbb{N}$ with $1 \leq j \leq m$;

3. $\mathrm{v}(\gamma_t) \neq \mathrm{v}(\gamma_j)$ for any $j \in \mathbb{N}$ with $1 \leq j \leq m$.

We consider each of these cases separately.

In the first case we have γ_t linearly independent but not strongly independent of $\{\gamma_1, \dots, \gamma_m\}$, so by the Exchange Lemma (lemma 8.2.10) we can find some element γ_{m+1} so that the set

$$\{\gamma_1, \dots, \gamma_m, \gamma_{m+1}\}$$

is strongly independent and with

$$\gamma_t \in \langle \gamma_1, \dots, \gamma_m, \gamma_{m+1} \rangle .$$

For this γ_{m+1} it is clear that one of the cases 2 or 3 above must hold. We therefore continue the construction according to these as set out below.

In the second case we have that γ_t is strongly independent of $\{\gamma_1, \dots, \gamma_m\}$ but $\mathrm{v}(\gamma_t) = \mathrm{v}(\gamma_j)$ for some $j \in \mathbb{N}$ with $1 \leq j \leq m$. Begin by setting $\gamma_{m+1} = \gamma_t$, and consider the element $r_j = \mathrm{st}\left(\frac{\gamma_j}{\gamma'_j}\right)$. By p.r.s.(1) and p.r.s.(2) we can find some γ'_{m+1} so that $\mathrm{st}\left(\frac{\gamma_{m+1}}{\gamma'_{m+1}}\right) = r_j$ with $\widetilde{\varrho}(\gamma'_{m+1}) = \widetilde{\varrho}(\gamma_{m+1})$. It is clear then that

$$\{\gamma_1, \dots, \gamma_{m+1}\} \quad \text{and} \quad \{\gamma'_1, \dots, \gamma'_{m+1}\}$$

satisfy the back-and-forth criteria.

In the third case, we have that $\mathrm{v}(\gamma_t) \neq \mathrm{v}(\gamma_j)$ for any $j \in \mathbb{N}$ with $1 \leq j \leq m$. Suppose that $\mathrm{v}(\gamma_{i_{j_1}}) < \mathrm{v}(\gamma_t) < \mathrm{v}(\gamma_{i_{j_2}})$, where either $\gamma_{i_{j_1}}$ or $\gamma_{i_{j_2}}$ (but not both) may not exist, in which case the obvious alterations must be made. We know that

$$\left(\mathrm{st}\left(\frac{\gamma_{i_1}}{\gamma'_{i_1}}\right), \dots, \mathrm{st}\left(\frac{\gamma_{i_{s'}}}{\gamma'_{i_{s'}}}\right)\right) = (r_{i_1}, \dots, r_{i_{s'}}) \in T_1,$$

so by the stQ-closure properties, there exists some r_t so that

$$(r_{i_1}, \dots, r_{i_{j_1}}, r_t, r_{i_{j_2}}, \dots, r_{i_{s'}}) \in T_1.$$

We therefore set $\gamma_{m+1} = \gamma_t$ and use p.r.s.(1) and p.r.s.(2) to find some element γ'_{m+1} so that $\widetilde{\varrho}(\gamma'_{m+1}) = \widetilde{\varrho}(\gamma_{m+1})$ and $\mathrm{st}\left(\frac{\gamma_{m+1}}{\gamma'_{m+1}}\right) = r_t$.

It is clear that the sets

$$\{\gamma_1, \ldots, \gamma_{m+1}\} \quad \text{and} \quad \{\gamma'_1, \ldots, \gamma'_{m+1}\}$$

satisfy the back-and-forth criteria.

This completes the back step. The forth step is practically identical.

Having completed the back-and-forth construction, we can produce a map

$$\widetilde{g}\colon \gamma_i \mapsto \gamma'_i \quad \text{for all } i \in \omega$$

which we lift to a map $g\colon \Gamma \to \Gamma$ using proposition 3.2.3.

It is clear that $g\colon x_i \mapsto y_i$ for $i \in \mathbb{N}$ and $1 \leq i \leq n$. We also claim that $g \in G_{T_1}$. To see this take any $n \in \omega$ and any $x_1, \ldots, x_n \in \Gamma$ with $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$. It is clear from the construction that

$$\left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right) \in T_1$$

and hence that $g \in G_{T_1}$, as required. ☐

Proposition 12.2.2. Suppose $T_1, T_2 \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\Gamma)_{>0})^n$ satisfy the stQ-closure properties. Suppose further that there exists some $(r_1, \ldots, r_n)$ with

$$(r_1, \ldots, r_n) \in T_1 \quad \text{but} \quad (r_1, \ldots, r_n) \notin T_2.$$

Then there exists a residue automorphism $g \in G$ such that $g \in G_{T_1}$ but $g \notin G_{T_2}$.

Proof. By p.r.s.(1) and p.r.s.(2) we can choose some $\bar{x}, \bar{y} \in \widetilde{\Gamma}^n$ with $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$, with $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ and so that

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) = (r_1, \ldots, r_n).$$

Using proposition 12.2.1 we can construct an automorphism such that $g\colon x_i \mapsto y_i$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$ and so that $g \in G_{T_1}$. But if we take $\gamma_1, \ldots, \gamma_n$ with $\mathrm{v}(\gamma_i) = \mathrm{v}(x_i)$ for $i \in \mathbb{N}$ with $1 \leq i \leq n$ then we know that

$$\left(\mathrm{st}\left(\frac{\gamma_1 g}{\gamma_1}\right), \ldots, \mathrm{st}\left(\frac{\gamma_n g}{\gamma_n}\right)\right) = (r_1, \ldots, r_n) \notin T_2.$$

Hence $g \notin G_{T_2}$ as required. ☐

Proposition 12.2.3. Suppose $T_1 \subseteq T_2$. Then $G_{T_1} \subseteq G_{T_2}$.

Proof. Let $h \in G_{T_1}$. Then for all $n \in \omega$ and all $x_1, \dots, x_n \in \Gamma$ with $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ we have that
$$\left(\mathrm{st}\left(\frac{x_1 h}{x_1}\right), \dots, \mathrm{st}\left(\frac{x_n h}{x_n}\right)\right) \in T_1.$$
But then
$$\left(\mathrm{st}\left(\frac{x_1 h}{x_1}\right), \dots, \mathrm{st}\left(\frac{x_n h}{x_n}\right)\right) \in T_2$$
and hence $h \in G_{T_2}$ as required. □

These results show us that we have a lattice of closed normal subgroups directly related to the lattice of stQ-closed subsets of $\bigcup_{n\in\omega}(\mathrm{stQ}(\Gamma)_{>0})^n$. We will make the correlation clearer by setting up a Galois correspondence in the next section. However, we wish to relate unions and intersections more precisely and therefore need a way of reducing arbitrary subsets of $\bigcup_{n\in\omega}(\mathrm{stQ}(\Gamma)_{>0})^n$ to stQ-closed subsets.

Definition 12.2.4. If $S = \bigcup_{n\in\omega} S_n \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\Gamma)_{>0})^n$ we define $\langle S \rangle$ to be the set
$$\langle S \rangle = \bigcup_{n\in\omega} \langle S_n \rangle$$
where $\langle S_n \rangle$ represents group closure in $\mathbb{R}^*$.

Definition 12.2.5. If $\bigcup_{n\in\omega} \bar{1} \subseteq S \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\Gamma)_{>0})^n$ we define the stQ**-reduction** of S as:
$$\overline{S}^{\mathrm{stQ}} = \bigcup_{\substack{T \subseteq \langle S \rangle \\ T \text{ stQ-closed}}} T.$$

We need now to check that this definition is well-defined, *i.e.* that $\overline{S}^{\mathrm{stQ}}$ satisfies the stQ-closure properties. In order to do this we require a further proposition.

Proposition 12.2.6. Suppose T_1 and T_2 are both stQ-closed. Then $\langle T_1 \cup T_2 \rangle = \{ t_1.t_2 : t_1 \in T_1, t_2 \in T_2, \mathrm{len}(t_1) = \mathrm{len}(t_2)\}$ is stQ-closed.

Proof. This follows directly from the fact that T_1 and T_2 are stQ-closed, and commutativity of $\mathbb{R}$. □

We now show that $\overline{S}^{\mathrm{stQ}}$ satisfies the stQ-closure properties.

Proposition 12.2.7. If $\bigcup_{n\in\omega} \bar{1} \subseteq S \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\Gamma)_{>0})^n$ then $\overline{S}^{\mathrm{stQ}}$ is stQ-closed and $\overline{S}^{\mathrm{stQ}} \subseteq \langle S \rangle$.

Proof. For the latter part we see from the definition that if $s \in \overline{S}^{\mathrm{stQ}}$ then $s \in T$ for some $T \subseteq \langle S \rangle$. It clearly follows that $s \in \langle S \rangle$ and hence that $\overline{S}^{\mathrm{stQ}} \subseteq \langle S \rangle$.

To show that $\overline{S}^{\mathrm{stQ}}$ is stQ-closed we will check the stQ-closure conditions 1–4.

For condition 1, suppose $t_1, t_2 \in \overline{S}^{\mathrm{stQ}}$ have the same length. We must show that $t_1.t_2 \in \overline{S}^{\mathrm{stQ}}$. But $t_1 \in T_1$ and $t_2 \in T_2$ for some $T_1, T_2 \subseteq \langle S \rangle$, both of which are stQ-closed. But by definition $\langle S \rangle$ is closed under inverses and products and so $\langle T_1 \cup T_2 \rangle \subseteq \langle S \rangle$. But by proposition 12.2.6 we know that $\langle T_1 \cup T_2 \rangle$ is stQ-closed and hence $\langle T_1 \cup T_2 \rangle \subseteq \overline{S}^{\mathrm{stQ}}$. It follows that $t_1.t_2 \in \overline{S}^{\mathrm{stQ}}$ as required.

The second condition follows similarly. Thus if we suppose $t_1 \in \overline{S}^{\mathrm{stQ}}$ then $t_1 \in T_1 \subseteq \langle S \rangle$ for some T_1 which is stQ-closed. Hence $t_1^{-1} \in T_1$ and so $t_1^{-1} \in \overline{S}^{\mathrm{stQ}}$ as required.

The final two conditions follow in an analogous manner. □

The next two results relate unions and intersections of the lattice between the two descriptions and given our aim are, perhaps, not very surprising. We use $\overline{H}$ to represent the topological closure of H in G.

Proposition 12.2.8. Let T_1 and T_2 be stQ-closed. Then

$$G_{\langle T_1 \cup T_2 \rangle} = \overline{\langle G_{T_1} \cup G_{T_2} \rangle}.$$

Proof. By proposition 12.2.3 and since $T_1, T_2 \subseteq \langle T_1 \cup T_2 \rangle$ we know that

$$G_{T_1}, G_{T_2} \subseteq G_{\langle T_1 \cup T_2 \rangle}\,.$$

But $G_{\langle T_1 \cup T_2 \rangle}$ is a closed group by theorem 11.4.3 and so

$$\overline{\langle G_{T_1} \cup G_{T_2} \rangle} \subseteq G_{\langle T_1 \cup T_2 \rangle}\,.$$

The reverse direction is slightly less straightforward. We suppose that $g \in G_{\langle T_1 \cup T_2 \rangle}$. We assume in order to find a contradiction that $g \notin \overline{\langle G_{T_1} \cup G_{T_2} \rangle}$. Since this latter set is closed we can find some basic open set $\mathscr{U}_{\bar{x}}$ for some $\bar{x} \in \Gamma$ so that $\mathscr{U}_{\bar{x}} = \{\, \alpha \in G : \bar{x}\alpha = \bar{x}g\}$ is disjoint from $\overline{\langle G_{T_1} \cup G_{T_2} \rangle}$. We may also assume that if $\bar{x} = (x_1, \ldots , x_n)$ then $x_1, \ldots , x_n$ are strongly independent.

We claim that for any $\bar{x} \in \Gamma$ we can find some $h \in \mathscr{U}_{\bar{x}}$ so that $h \in \overline{\langle G_{T_1} \cup G_{T_2} \rangle}$.

So consider $x_1, \ldots , x_n$. Let $m \leq n$ and $i_1, \ldots , i_m \in \{1, \ldots , n\}$ be such that $\mathrm{v}(x_{i_1}) < \cdots < \mathrm{v}(x_{i_m})$. We may also suppose that for any x_j where $1 \leq j \leq n$ there is some $k \in \mathbb{N}$ with $1 \leq k \leq m$ so that $\mathrm{v}(x_j) = \mathrm{v}(x_{i_k})$. Now let

$$\left(\mathrm{st}\left(\frac{x_{i_1} g}{x_{i_1}}\right), \ldots , \mathrm{st}\left(\frac{x_{i_m} g}{x_{i_m}}\right)\right) = (r_1, \ldots , r_m).$$

Clearly we must have $(r_1, \dots, r_m) \in \langle T_1 \cup T_2 \rangle$ since $g \in G_{\langle T_1 \cup T_2 \rangle}$. In particular, then, $(r_1, \dots, r_m) = (s_1, \dots, s_m) \cdot (s'_1, \dots, s'_m)$ for some $(s_1, \dots, s_m) \in T_1$ and $(s'_1, \dots, s'_m) \in T_2$.

Let $h_1 \in G_{T_1}$ be any map such that

$$\left(\mathrm{st}\left(\frac{x_{i_1} h_1}{x_{i_1}}\right), \dots, \mathrm{st}\left(\frac{x_{i_m} h_1}{x_{i_m}}\right) \right) = (s_1, \dots, s_m).$$

We know that such a residue automorphism exists by proposition 12.2.1.

We now claim that there is some residue automorphism $h_2 \in G_{T_2}$ such that

$$h_2 \colon x_j h_1 \mapsto x_j g \qquad \text{for } 1 \leq j \leq n.$$

To show this we simply need to note that $\varrho(x_j h_1) = \varrho(x_j g)$, that

$$\{x_1 h_1, \dots, x_n h_1\} \quad \text{and} \quad \{x_1 g, \dots, x_n g\}$$

are both strongly independent sets and that

$$\mathrm{st}\left(\frac{x_{j_1} h_1}{x_{j_2} h_1}\right) = \mathrm{st}\left(\frac{x_{j_1}}{x_{j_2}}\right) = \mathrm{st}\left(\frac{x_{j_1} g}{x_{j_2} g}\right)$$

for all $j_1, j_2 \in \mathbb{N}$ with $1 \leq j_1, j_2 \leq n$. It follows by theorem 10.1.1 that we can find some $h_2 \in G_{\mathrm{v}}$ of the required sort. Moreover, consider $\left(\mathrm{st}\left(\frac{x_{i_1} h_2}{x_{i_1}}\right), \dots, \mathrm{st}\left(\frac{x_{i_m} h_2}{x_{i_m}}\right) \right)$. Since h_1 and g both preserve values, we have

$$\begin{aligned}
\left(\mathrm{st}\left(\frac{x_{i_1} h_2}{x_{i_1}}\right), \dots, \mathrm{st}\left(\frac{x_{i_m} h_2}{x_{i_m}}\right) \right) &= \left(\mathrm{st}\left(\frac{x_{i_1} h_1 h_2}{x_{i_1} h_1}\right), \dots, \mathrm{st}\left(\frac{x_{i_m} h_1 h_2}{x_{i_m} h_1}\right) \right) \\
&= \left(\mathrm{st}\left(\frac{x_{i_1} g}{x_{i_1} h_1}\right), \dots, \mathrm{st}\left(\frac{x_{i_m} g}{x_{i_m} h_1}\right) \right) \\
&= \left(\mathrm{st}\left(\frac{x_{i_1} g h^{-1}}{x_{i_1}}\right), \dots, \mathrm{st}\left(\frac{x_{i_m} g h^{-1}}{x_{i_m}}\right) \right) \\
&= (r_1, \dots, r_m) \cdot (s_1, \dots, s_m)^{-1} \\
&= (s'_1, \dots, s'_m).
\end{aligned}$$

We can therefore use stQ-closure property 4 in conjunction with proposition 10.1.4 to ensure that $h_2 \in G_{T_2}$.

But in this case $h_1 \in G_{T_1}$ and $h_2 \in G_{T_2}$ and $\bar{x} h_1 h_2 = \bar{x} g$. Thus $h_1 h_2 \in G_{\langle T_1 \cup T_2 \rangle}$ where $h_1 \in G_{T_1}$ and $h_2 \in G_{T_2}$. But in this case

$$h_1 h_2 \in \overline{\langle G_{T_1} \cup G_{T_2} \rangle}$$

and so

$$\mathscr{U}_{\bar{x}} \cap \overline{\langle G_{T_1} \cup G_{T_2} \rangle} \neq \emptyset.$$

Since this holds for any $\bar{x}$ of finite length, this contradicts the closure of $\overline{\langle G_{T_1} \cup G_{T_2} \rangle}$. We must therefore have $g \in \overline{\langle G_{T_1} \cup G_{T_2} \rangle}$ as required. $\square$

Proposition 12.2.9. Let T_1 and T_2 be stQ-closed. Then

$$G_{\overline{T_1 \cap T_2}^{\mathrm{stQ}}} = G_{T_1} \cap G_{T_2}\,.$$

Proof. We first show that $G_{\overline{T_1 \cap T_2}^{\mathrm{stQ}}} \subseteq G_{T_1} \cap G_{T_2}$. To do this we note that if T_1 and T_2 are stQ-closed, then $T_1 \cap T_2 = \langle T_1 \cap T_2 \rangle$. For suppose $s_1, s_2 \in T_1 \cap T_2$, then $s_1.s_2 \in T_1$ and $s_1.s_2 \in T_2$ and so $s_1.s_2 \in T_1 \cap T_2$. Similarly for s_1^{-1}. But in this case it follows that $\overline{T_1 \cap T_2}^{\mathrm{stQ}} \subseteq T_1 \cap T_2$.

Now if $g \in G_{\overline{T_1 \cap T_2}^{\mathrm{stQ}}}$ then for $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ it is clear that

$$\left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right) \in \overline{T_1 \cap T_2}^{\mathrm{stQ}} \subseteq T_1 \cap T_2.$$

In particular, then, it is contained in both T_1 and T_2. But the $x_1, \ldots, x_n$ were chosen arbitrarily and hence $g \in G_{T_1}$ and $g \in G_{T_2}$. The result follows.

Now for the reverse direction, suppose that $g \in G_{T_1} \cap G_{T_2}$ and let

$$T = \left\{ \left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right) : n \in \omega, x_1, \ldots, x_n \in \Gamma \text{ and } \mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n) \right\}$$

be the set of standard part tuples associated with it.

Since $g \in G_{T_1}$ and $g \in G_{T_2}$ it is clear that $T \subseteq T_1$ and $T \subseteq T_2$. But T_1, T_2 are both stQ-closed, so $\overline{T}^{\mathrm{stQ}} \subseteq T_1$ and $\overline{T}^{\mathrm{stQ}} \subseteq T_2$. So $\overline{T}^{\mathrm{stQ}} \subseteq T_1 \cap T_2$ and since $\overline{T}^{\mathrm{stQ}}$ is clearly stQ-closed we have $\overline{T}^{\mathrm{stQ}} \subseteq \overline{T_1 \cap T_2}^{\mathrm{stQ}}$. It follows that $g \in G_{\overline{T_1 \cap T_2}^{\mathrm{stQ}}}$ as required. $\square$

The results we have concerning the lattice structure of the closed normal subgroups can be used to provide more general information about the nature of the automorphism group. As an example of this, we give the proposition 12.2.11 below.

Lemma 12.2.10. Suppose $S = \{p_1, \ldots, p_n\}$ is a set of primes and that $p \notin S$. Then $\langle S \rangle \neq \langle S \cup \{p\} \rangle$ where

$$\langle S \rangle = \{\, s \in \mathbb{Q} : n \in \omega, x_1, \ldots, x_n \in S, l_1, \ldots, l_n \in \mathbb{Z}, s = x_1^{l_1} \ldots x_n^{l_n} \}.$$

Proof. Clearly $p \in \langle S \cup \{p\} \rangle$. We claim that $p \notin \langle S \rangle$. For suppose otherwise, then $p = x_1^{l_1} \dots x_n^{l_n}$ for some $x_1, \dots, x_n \in S$ and $l_1, \dots, l_n \in \mathbb{Z}$. But some of $l_1, \dots, l_n$ may be negative, and without loss of generality we may assume that $l_1, \dots, l_m$ are negative where $m \leq n$. Hence

$$x_1^{-l_1} \dots x_m^{-l_m} p = x_{m+1}^{l_{m+1}} \dots x_n^{l_n}.$$

It follows immediately by unique factorisation that $p = x_k$ for some $k \in \mathbb{N}$ with $m + 1 \leq k \leq n$, contradicting the assumption that $p \notin S$. $\square$

Proposition 12.2.11. Let Γ be a countable pseudo-recursively saturated model of Presburger arithmetic with trivial centre. Then Γ has $2^{\aleph_0}$ closed normal subgroups.

Proof. From the earlier work on closed normal subgroups we know that it suffices to show there are $2^{\aleph_0}$ distinct stQ-closed subsets of $\bigcup_{n \in \omega} (\mathrm{stQ}(\Gamma)_{>0})^n$. Clearly there cannot be more than this number as Γ is countable, so we need only show that there are at least this many. Now for any model Γ we know that $\mathbb{Q} \in \mathrm{stQ}(\Gamma)$, so we will concentrate on the stQ-closed sets generated from sets of tuples of identical primes. Let

$$P(p) = \{(p^{l_1}, \dots, p^{l_n}) \in \mathbb{Q}^n : n \in \omega, l_1, \dots, l_n \in \mathbb{Z}\}.$$

We claim that if S is a set of tuples of primes taken as a union of these $P(p)$ and $(q) \notin S$ then $\overline{S}^{\mathrm{stQ}} \neq \overline{S \cup P(q)}^{\mathrm{stQ}}$. We do this by showing that the 1-tuple $(q) \in \overline{S \cup P(q)}^{\mathrm{stQ}}$ whilst $(q) \notin \overline{S}^{\mathrm{stQ}}$.

Now clearly $\bigcup_{n \in \omega} Q_n$ is stQ-closed where

$$Q_n = \{(q^{l_1}, \dots, q^{l_n}) \in \mathbb{Q}^n : l_1, \dots, l_n \in \mathbb{Z}\}.$$

But then $(q) \in \overline{S \cup P(q)}^{\mathrm{stQ}}$, whilst by the previous lemma (12.2.10) we know that $(q) \notin \overline{S}^{\mathrm{stQ}}$. The result follows immediately from this. $\square$

12.3 A pair of Galois connections

Definition 12.3.1. Let $\Gamma_{\mathrm{v}} \subseteq \bigcup_{n \in \omega} \Gamma^n$ be the set of tuples $\bar{x} = (x_1, \dots, x_n) \in \Gamma^n$ such that $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$.

We can extend this definition to $\widetilde{\Gamma}_{\mathrm{v}}$ in the usual way.

Definition 12.3.2. Suppose $T \subseteq \bigcup_{n\in\omega}(\mathrm{stQ}(\Gamma)_{>0})^n$ is stQ-closed and $\bar{x}, \bar{y} \in \widetilde{\Gamma}_{\mathrm{v}}$. Then we say that $\bar{x} \sim_T \bar{y}$ if $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ and

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) \in T.$$

This is an equivalence relation on Γ_{v} by the stQ-closure conditions on T, as the following proposition shows.

Proposition 12.3.3. The relation $\sim_T$ is an equivalence relation.

Proof. For reflexivity, note that $\overbrace{(1, \ldots, 1)}^{n} \in T$ for all $n \in \mathbb{N}$. But if $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ then clearly $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{x})$ and

$$\left(\mathrm{st}\left(\frac{x_1}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{x_n}\right)\right) = (1, \ldots, 1) \in T.$$

Hence $\bar{x} \sim_T \bar{x}$ as required.

For symmetry, if $\bar{x} \sim_T \bar{y}$ then $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ and

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) \in T.$$

But by the second stQ-closure property of T we know that

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right)^{-1} = \left(\mathrm{st}\left(\frac{y_1}{x_1}\right), \ldots, \mathrm{st}\left(\frac{y_n}{x_n}\right)\right) \in T,$$

and hence $\bar{y} \sim_T \bar{x}$.

For transitivity, suppose $\bar{x} \sim_T \bar{y}$ and $\bar{y} \sim_T \bar{z}$. Clearly $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y}) = \mathrm{tp}(\bar{z})$ and since both

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) \text{ and } \left(\mathrm{st}\left(\frac{y_1}{z_1}\right), \ldots, \mathrm{st}\left(\frac{y_n}{z_n}\right)\right)$$

are in T, we know by the first stQ-closure property of T that

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right) \cdot \mathrm{st}\left(\frac{y_1}{z_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right) \cdot \mathrm{st}\left(\frac{y_n}{z_n}\right)\right) = \left(\mathrm{st}\left(\frac{x_1}{z_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{z_n}\right)\right) \in T.$$

Hence $\bar{x} \sim_T \bar{z}$ as required. □

Lemma 12.3.4. Suppose $\bar{x}, \bar{y} \in \widetilde{\Gamma}_{\mathrm{v}}$. Then $\bar{x} \sim_T \bar{y}$ if and only if $\bar{x}g = \bar{y}$ for some $g \in G_T$.

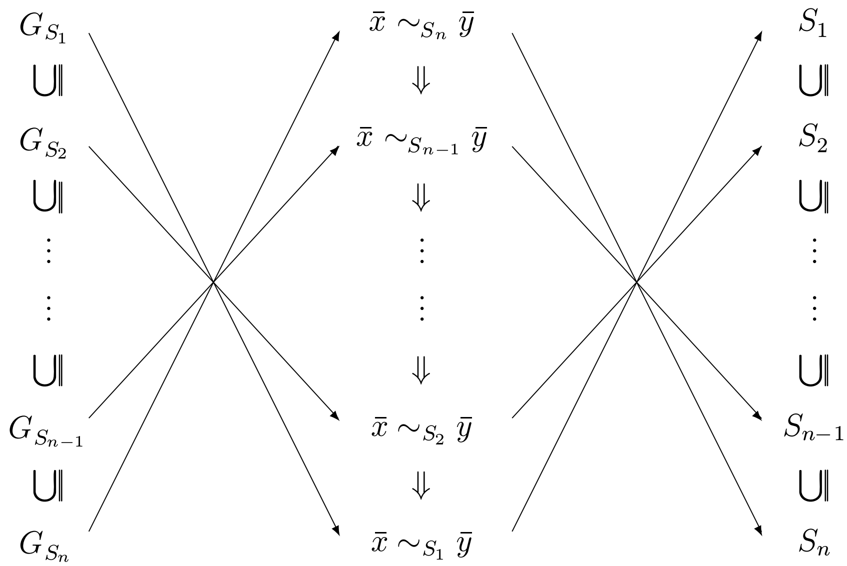

Figure 12.1: A Pair of Galois Connections.

Proof. First we suppose that $\bar{x} \sim_T \bar{y}$. Then

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \dots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) \in T,$$

and by proposition 12.2.1 there exists an automorphism $g \in G_T$ such that $\bar{x}g = \bar{y}$, as required.

Now suppose that $\bar{x}g = \bar{y}$ for some $g \in G_T$. Then, again, by the definition of G_T we know that

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \dots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) \in T,$$

and hence that $\bar{x} \sim_T \bar{y}$ as required. □

It follows from the above lemma 12.3.4 that we could equally have defined $\bar{x} \sim_T \bar{y}$ to hold if and only if $\bar{x}g = \bar{y}$ for some $g \in G_T$.

Figure 12.1 represents the Galois connections. Note that in the representation the inclusions and implications appear to be linearly ordered. This has been done purely for illustrative convenience, as in general the order is only partial.

The next theorem shows that the diagram does indeed represent a Galois connection.

Theorem 12.3.5. Suppose Γ is a countable, pseudo-recursively saturated model of Presburger arithmetic and that $T_1, T_2 \in \bigcup_{n \in \omega} (\mathrm{stQ}(\Gamma)_{>0})^n$ are both stQ-closed. Then

1. $G_{T_1} \subseteq G_{T_2}$ if and only if for all $\bar{x}, \bar{y} \in \widetilde{\Gamma}_{\mathrm{v}}$ we have $\bar{x} \sim_{T_1} \bar{y} \quad \Rightarrow \quad \bar{x} \sim_{T_2} \bar{y}$.

2. The arrows are bijections.

Proof. For the first claim, suppose that $G_{T_1} \subseteq G_{T_2}$. Now if $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ and $\bar{x} \sim_{T_1} \bar{y}$ then there is some $g \in G_{T_1}$ such that $\bar{x}g = \bar{y}$. But then $g \in G_{T_2}$ and hence $\bar{x} \sim_{T_2} \bar{y}$ as required.

Now suppose that for all $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ we have that $\bar{x} \sim_{T_1} \bar{y} \quad \Rightarrow \quad \bar{x} \sim_{T_2} \bar{y}$. Consider $g_1 \in G_{T_1}$. Then for any $\bar{x}, \bar{y} \in \Gamma^n$ such that $\bar{x}g_1 = \bar{y}$ we know that there is some $g_2 \in G_{T_2}$ such that $\bar{x}g_2 = \bar{y}$. In other words

$$\left(\mathrm{st}\left(\frac{x_1 g_1}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g_1}{x_n}\right)\right) \in T_2$$

for every $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$. Hence by the definition of G_{T_2} we know that $g_1 \in G_{T_2}$ as required.

Indeed the second claim follows directly from this, however we also provide an explicit proof of this in the hope that it will provide some insight in to the nature of the bijections.

So we aim to show that

$$\{ g \in G : \forall\, \bar{x}, \bar{y}, \bar{x} \sim_{T_1} \bar{y} \quad \Rightarrow \quad \bar{x}g \sim_{T_1} \bar{y}\} = G_{T_1}\,.$$

So suppose that $g \in G_{T_1}$ and $\bar{x} \sim_{T_1} \bar{y}$. Then

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) \in T_1$$

and

$$\left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right) \in T_1.$$

So by the first stQ-closure property we know that

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right) \cdot \mathrm{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right) \cdot \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right)$$
$$= \left(\mathrm{st}\left(\frac{x_1 g}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g}{y_n}\right)\right) \in T_1.$$

Hence $\bar{x}g \sim_{T_1} \bar{y}$ as required.

Now suppose that $g \in \{ g \in G : \forall\, \bar{x}, \bar{y}, \bar{x} \sim_{T_1} \bar{y} \quad \Rightarrow \quad \bar{x}g \sim_{T_1} \bar{y}\}$. Then if $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ we know that $\bar{x} \sim_{T_1} \bar{x}$. Hence $\bar{x}g \sim_{T_1} \bar{x}$ and so

$$\left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right) \in T_1.$$

As $x_1, \dots, x_n$ were chosen arbitrarily in Γ_{v}, this must therefore hold for all such tuples, and hence by the definition of G_{T_1} we know that $g \in G_{T_1}$ as required. □

Finally, we would like to find a set of criteria which will allow us to distinguish the relations $\sim_T$ from arbitrary G-invariant equivalence relations. These are given in the next result.

Proposition 12.3.6. Suppose Γ is countable pseudo-recursively saturated and that $\sim$ is a G-invariant equivalence relation on $\widetilde{\Gamma}_{\mathrm{v}}$. Suppose further that

1. if $x_1, \dots, x_n \sim y_1, \dots, y_n$ and $m \leq n$ then
$$x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_n \sim y_1, \dots, y_{m-1}, y_{m+1}, \dots, y_n;$$
2. if $x_1, \dots, x_n \sim y_1, \dots, y_n$ and $m \leq n+1$ then there is at least one pair x'_m, y'_m with
$$x_1, \dots, x_{m-1}, x'_m, x_m, \dots, x_n \sim y_1, \dots, y_{m-1}, y'_m, y_m, \dots, y_n;$$
3. if $\bar{x}, \bar{y}$ are such that $\bar{x} \frown \bar{y}$ then $\bar{x} \sim \bar{y}$.

Then there exists some stQ-closed T such that $\bar{x} \sim \bar{y}$ if and only if $\bar{x} \sim_T \bar{y}$.

Proof. We begin by noting that the third criterion in conjunction with the condition that $\sim$ is G-invariant is equivalent to saying:

3. if $\bar{x} \sim \bar{y}$ then for every $\bar{z}, \bar{w}$ with
$$\left(\mathrm{st}\left(\frac{z_1}{w_1}\right), \dots, \mathrm{st}\left(\frac{z_n}{w_n}\right)\right) = \left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \dots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right)$$
we must have $\bar{z} \sim \bar{w}$.

One direction of this equivalence is obvious. For the other suppose that $\sim$ is G-invariant so that if $\bar{x} \sim \bar{y}$ then $\bar{x}g \sim \bar{y}g$ for all $g \in G$ and that it satisfies criterion 3. Take any $\bar{x}, \bar{y} \in \widetilde{\Gamma}_{\mathrm{v}}$ so that $\bar{x} \sim \bar{y}$ and let $\bar{z}, \bar{w} \in \widetilde{\Gamma}_{\mathrm{v}}$ be such that
$$\left(\mathrm{st}\left(\frac{z_1}{w_1}\right), \dots, \mathrm{st}\left(\frac{z_n}{w_n}\right)\right) = \left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \dots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right).$$
By p.r.s.(1) we can find $\bar{z}'$ such that $\widetilde{\varrho}(z'_i) = \widetilde{\varrho}(z_i)$ for all i with $\bar{z}' \frown \bar{z}$ and since $\mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n)$ we can construct a residue automorphism $g \in G$ such that

$g\colon \bar{x} \mapsto \bar{z}'$. But then $\bar{x}g = \bar{z}' \frown \bar{z}$ so by the criterion we know that $\bar{x}g \sim \bar{z}$. We also know that

$$\mathrm{st}\left(\frac{y_i g}{w_i}\right) = \mathrm{st}\left(\frac{y_i g}{x_i g}\right) \cdot \mathrm{st}\left(\frac{x_i g}{z_i}\right) \cdot \mathrm{st}\left(\frac{z_i}{w_i}\right) = \mathrm{st}\left(\frac{y_i}{x_i}\right) \cdot \mathrm{st}\left(\frac{z_i'}{z_i}\right) \cdot \mathrm{st}\left(\frac{z_i}{w_i}\right) = 1$$

and hence that $\bar{y}g \frown \bar{w}$. It follows that $\bar{y}g \sim \bar{w}$ and hence by transitivity $\bar{z} \sim \bar{w}$ as required.

Having established this we can now define the T which we will be using. So suppose that $\sim$ is an equivalence relation on Γ_{v} satisfying the criteria given in the statement. Then we define

$$S_n = \left\{ \left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) : \bar{x}, \bar{y} \in \widetilde{\Gamma}, \mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n), \bar{x} \sim \bar{y} \right\},$$

so that $T = \bigcup_{n\in\omega} S_n$. We must show that T is stQ-closed. We will consider each of the stQ conditions in turn.

1. For each $\bar{x} \in \Gamma_{\mathrm{v}}$ we know that $\bar{x} \sim \bar{x}$ and by p.r.s.(3) we know that we can find such tuples of arbitrary length. Hence S_n is non-empty for all $n \in \omega$. If $\bar{r} \in S_n$ then
$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) = \bar{r}$$
for some $\bar{x}, \bar{y} \in \widetilde{\Gamma}_{\mathrm{v}}$ with $\bar{x} \sim \bar{y}$. So suppose $\bar{r}, \bar{s} \in S_n$. Let $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ be such that
$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) = \bar{r} \text{ and } \left(\mathrm{st}\left(\frac{z_1}{w_1}\right), \ldots, \mathrm{st}\left(\frac{z_n}{w_n}\right)\right) = \bar{s}$$
with $\bar{x} \sim \bar{y}$ and $\bar{z} \sim \bar{w}$. Then by p.r.s.(2) we can find some $\bar{z}'$ such that
$$\left(\mathrm{st}\left(\frac{y_1}{z_1'}\right), \ldots, \mathrm{st}\left(\frac{y_n}{z_n'}\right)\right) = \left(\mathrm{st}\left(\frac{z_1}{w_1}\right), \ldots, \mathrm{st}\left(\frac{z_n}{w_n}\right)\right) = \bar{s}.$$
By the variation of criterion 3 above we know that $\bar{y} \sim \bar{z}'$ and so by transitivity $\bar{x} \sim \bar{z}'$. Hence
$$\left(\mathrm{st}\left(\frac{x_1}{z_1'}\right), \ldots, \mathrm{st}\left(\frac{x_n}{z_n'}\right)\right) = \bar{r}.\bar{s} \in S_n$$
as required.

2. Suppose $\bar{r} \in S_n$. Then there exist $\bar{x} \sim \bar{y}$ with
$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) = \bar{r} \in S_n.$$
But then by symmetry $\bar{y} \sim \bar{x}$ and hence
$$\left(\mathrm{st}\left(\frac{y_1}{x_1}\right), \ldots, \mathrm{st}\left(\frac{y_n}{x_n}\right)\right) = \bar{r}^{-1} \in S_n.$$

3. This follows immediately by criterion 1 above.

4. This also follows immediately, by criterion 2.

The set T is therefore stQ-closed as required. We must now show that for all $\bar{x}, \bar{y} \in \widetilde{\Gamma}_{\mathrm{v}}$ we have $\bar{x} \sim \bar{y}$ if and only if $\bar{x} \sim_T \bar{y}$. It is immediately clear that if $\bar{x} \sim \bar{y}$ then $\bar{x} \sim_T \bar{y}$. For the reverse direction suppose that $\bar{x} \sim_T \bar{y}$. Then

$$\left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right) \in S_n$$

so for some $\bar{z}, \bar{w}$ with $\bar{z} \sim \bar{w}$ it must be the case that

$$\left(\mathrm{st}\left(\frac{w_1}{z_1}\right), \ldots, \mathrm{st}\left(\frac{w_n}{z_n}\right)\right) = \left(\mathrm{st}\left(\frac{x_1}{y_1}\right), \ldots, \mathrm{st}\left(\frac{x_n}{y_n}\right)\right).$$

But then by the alternative version of criterion 3 we therefore have that $\bar{x} \sim \bar{y}$ as required. □

12.4 The relevance of valuation order

In previous sections the majority of the results have concerned elements from Γ of increasing values. In many cases it is clearer to work using elements ordered in this way since it clarifies the manner in which the automorphisms are acting on the elements. It turns out, however, that choosing elements in this way is not necessary for the results to hold and we can remove this requirement from the definitions.

In particular, the closed normal subgroups of G were represented as subgroups of the form

$$\begin{aligned} G_S &= \left\{ g \in G_{\mathrm{v}} : \forall n \in \omega \, \forall \, \mathrm{v}(x_1) < \cdots < \mathrm{v}(x_n) \left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right) \in S \right\} \\ &= \left\{ g \in G_{\mathrm{v}} : \forall n \in \omega \, \forall (x_1, \ldots x_n) \in \Gamma_{\mathrm{v}} \left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right) \in S \right\}. \end{aligned}$$

This can therefore be written in the mathematically simpler form of

$$G_S = \left\{ g \in G_{\mathrm{v}} : \forall n \in \omega \, \forall \{\mathrm{v}(x_1), \ldots, \mathrm{v}(x_n)\} \left(\mathrm{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathrm{st}\left(\frac{x_n g}{x_n}\right)\right) \in S \right\}.$$

It is by no means clear that these two definitions are equivalent, but by referring to previous results and definitions we can show this to be the case quite easily.

In proposition 10.4.4 care has been taken to ensure that the ordering of the elements and the standard parts taken from $\mathrm{stQ}(h)$ do not effect the validity of the result. It is this proposition which is used to ensure that we can map elements close to each other, given the requirements of the proposition are satisfied. Working through the results of the previous chapter, then, we see that any reference to the ordering of the elements can be avoided by judicious use of proposition 10.4.4. In particular its use in 11.5.10 (fourth paragraph from the end of the proof) can be applied without the restriction of the ordering.

The result is a surprising one, which can be seen by considering the stQ-closed sets as the images of projections of $\mathrm{stQ}^{\mathrm{V}}$, as described in section 10.3. In this case the ordering of the values is reflected in the tuples from $\mathrm{stQ}(\Gamma)$, since the element from $\mathrm{stQ}^{\mathrm{V}}$ is a coloured set of ordered values with the ordering being preserved by the projection.

We will now give a strict proof using the elements of the stQ-closed set S from which G_S is derived.

Proposition 12.4.1. Suppose S is stQ-closed, $1 \leq m \leq n-1$, and that

$$(r_1, \dots, r_{m-1}, \quad r_m, \quad r_{m+1}, \; r_{m+2}, \dots, r_n) \in S.$$

Then

$$(r_1, \dots, r_{m-1}, \; r_{m+1}, \quad r_m, \quad r_{m+2}, \dots, r_n) \in S.$$

Proof. We will make extensive use of the stQ-closure properties of S. Let

$$a_1 = (r_1, \dots, r_{m-1}, r_m, r_{m+1}, r_{m+2}, \dots, r_n).$$

Then by the fourth stQ-closure property applied to a_1, we have that for some $s, t \in \mathrm{stQ}(\Gamma)$ the element

$$(r_1, \dots, r_{m-1}, s, r_m, r_{m+1}, t, r_{m+2}, \dots, r_n)$$

is in S. It follows by the second and third stQ-closure properties that the following are all in S:

$$\begin{array}{lllllllll} a_1 = (& r_1, & \dots, & r_{m-1}, & r_m, & r_{m+1}, & r_{m+2}, & \dots, & r_n &), \\ a_2 = (& r_1, & \dots, & r_{m-1}, & s, & r_m, & r_{m+2}, & \dots, & r_n &), \\ a_3 = (& r_1^{-1}, & \dots, & r_{m-1}^{-1}, & r_m^{-1}, & t^{-1}, & r_{m+2}^{-1}, & \dots, & r_n^{-1} &), \\ a_4 = (& r_1, & \dots, & r_{m-1}, & r_{m+1}, & t, & r_{m+2}, & \dots, & r_n &), \\ a_5 = (& r_1^{-1}, & \dots, & r_{m-1}^{-1}, & s^{-1}, & r_{m+1}^{-1}, & r_{m+2}^{-1}, & \dots, & r_n^{-1} &), \end{array}$$

(where a_1 has been repeated). It is clear that all five of the tuples, $a_1, \dots, a_5$, have length n. But now by the first stQ-closure property we know that $a_1 \dots a_5 \in S$ where the operation is multiplication of the reals applied pointwise. Calculating this we get

$$a_1 \dots a_5 = (\; r_1, \; \dots, \; r_{m-1}, \; r_{m+1}, \; r_m, \; r_{m+2}, \; \dots, \; r_n \;)$$

and hence the required element is contained in S. □

Since any permutation can be expressed as a product of transpositions (see for example Rotman [49, theorem 3.4] or any other introduction to group theory), it follows that if $(r_1, \dots, r_n) \in S$ then any permutation of the r_i's will also be contained in S.

Proposition 12.4.2. Suppose S is stQ-closed, $1 \leq m \leq n-1$, and that

$$(r_1, \dots, r_{m-1}, \quad r_m, \quad r_{m+1}, \dots, r_n) \in S.$$

Then

$$(r_1, \dots, r_{m-1}, \; r_m, \; r_m, \; r_{m+1}, \dots, r_n) \in S.$$

Proof. The proof follows similar lines to that of proposition 12.4.1. By the fourth stQ-closure property applied to a_1 (as found in proposition 12.4.1), we have that for some $s, t \in \mathrm{stQ}(\Gamma)$ the element

$$(r_1, \dots, r_{m-1}, s, r_m, t, r_{m+1}, \dots, r_n)$$

is in S. It follows by the second and third stQ-closure properties that the following are also in S:

$$a_1 = (\; r_1, \; \ldots, \; r_{m-1}, \; s, \; r_m, \; r_{m+1}, \; \ldots, \; r_n \;),$$

$$a_2 = (\; r_1^{-1}, \; \ldots, \; r_{m-1}^{-1}, \; s^{-1}, \; t^{-1}, \; r_{m+1}^{-1}, \; \ldots, \; r_n^{-1} \;),$$

$$a_3 = (\; r_1, \; \ldots, \; r_{m-1}, \; r_m, \; t, \; r_{m+1}, \; \ldots, \; r_n \;).$$

It is clear that all three of the tuples, $a_1, \ldots, a_3$, have length $n+1$. But now by the first stQ-closure property we know that $a_1 \ldots a_3 \in S$ where the operation is multiplication of the reals applied pointwise. Calculating this we get

$$a_1 \ldots a_3 = (\; r_1, \; \ldots, \; r_{m-1}, \; r_m, \; r_m, \; r_{m+1}, \; \ldots, \; r_n \;)$$

and hence the required element is contained in S. □

It follows from the above two propositions that we are able to permute and duplicate elements in the tuple and remain inside S. In particular, then, the two definitions of the set G_S given above are equivalent, since the ordering of the values becomes irrelevant.

Chapter 13

Conclusion

13.1 Overview

The purpose behind much of this thesis has been twofold. On the one hand we have attempted to consider essentially well known results about Presburger arithmetic from a new perspective and on the other we have hoped to introduce new, as yet unpublished ideas which help to give a more complete description of the theory of Presburger arithmetic, in particular with the hope of describing the automorphisms of a certain class of Presburger groups.

To partition in a rather crude manner, the first of these objectives has been tackled in the initial six chapters, where various important results are proven. A number of useful techniques were introduced in the first few chapters, notably with an analysis of residues, and the relationship between Presburger groups and a certain sort of divisible ordered abelian group. Chapter 4 dealt with the mathematically and historically important result of quantifier elimination for Presburger groups which also allows us to exhibit the completeness of the theory. We then went on to discuss the topology surrounding the space of types, and the compactness of this space is looked at in chapter 6. In chapter 6 we also provided a proof due to Kaye that $\widehat{\mathbb{Z}}$ can be considered to be a model of Presburger arithmetic. In keeping with the preceding chapters, our discussion up to this point was taken from a more algebraic perspective than would normally be the case. Although generally the results given are not new (excepting in particular the consideration of $\widehat{\mathbb{Z}}$ as a Presburger group), their presentation would usually be aimed towards model theorists and logicians. An attempt was therefore made to avoid the use of model theoretic terminology and to present the results in a way more amenable to a general mathematician. It was also hoped that such an analysis would provide the opportunity to explore new areas which might previously have been overlooked. Much of the work in these earlier chapters can be attributed to Richard Kaye, who presented a Birmingham University Study Group on the subject.

The second of the intended objectives was to present new results especially in relation to embeddings and automorphisms. Chapter 8 introduced new terminology with the aim of describing a new class of models of Presburger arithmetic, namely the pseudo-recursively saturated models, which are suitably saturated with elements of certain types in order to ensure that abundant automorphisms exist for them. When the automorphism groups of models of Presburger arithmetic were originally considered it was believed that recursive saturation would be necessary in order to study them effectively. It is surprising therefore that the considerably weaker notion of pseudo-recursive saturation suffices to provide the structure needed to allow automorphisms to be generated in a relatively straightforward manner. The key lies in the relationship between pseudo-recursively saturation and homogeneity. Both pseudo-recursive saturation and this relationship were looked at in chapter 9.

We then turned our attention towards normal subgroups and spent some time in chapter 10 proving results relating to conjugates of automorphisms of countable pseudo-recursively saturated models of Presburger arithmetic. These split nicely into the two cases of value-defying and value-preserving automorphisms. This led on, at the conclusion of chapter 11, to a complete correlation between the closed normal subgroups of the automorphism group and a particular type of stQ-closed set deriving from the set of standard parts of a particular model. Also in this chapter we looked at some important examples of closed normal subgroups of this type.

By using these results and others we could examine more closely the structure of the set of closed normal subgroups of our automorphism group and we did this in chapter 12, along with some observations about this structure, in particular we set up a pair of Galois connections between the various structures.

At the end of chapter 12 the especially interesting observation was made that, whilst instructive, the valuation order of elements were not actually relevant when looking at the relationship between closed normal subgroups and stQ-closed sets. This was despite the inclusion of restrictions in many of the results up to this point which required valuation orders to be considered. Referring back to previous chapters, however, it became clear that this was a reasonable ascertainment.

13.2 Further development

There are a number of obvious ways in which the work in this thesis could be developed further or extended. Although the relationship for closed normal subgroups appears in some sense complete, questions arise as to what can be discovered from the information which these results provide. There is also the obvious equivalent question for normal subgroups which are not necessarily closed. Existing results can go some way towards answering this question and suggest ways that the question might be tackled, for example by viewing the results in terms of automorphisms of the coloured sets of ordered values. The crucial difference lies in the need to consider tuples of only finite length in the closed case, whereas when considering all normal subgroups infinite tuples are needed with the consequent difficulties involved. However, it is clear that all of the information is contained within the description in terms of coloured sets of ordered values, where the length of tuples only becomes a consideration when we begin to apply projections. This may, then, provide the means to answering questions concerning the nature of the normal subgroups of our automorphism groups, but clearly more work would be needed to produce results in this area.

Another question which emerges from this work concerns the composition series for the automorphism group. For example, we may ask what the normal or closed normal subgroups of G_{v} are. Again, existing work gives us partial results pertaining to this. By reference to the proof it is clear that proposition 10.4.4 will not hold and that for automorphisms $g_1, \dots, g_n \in G_{\mathrm{v}}$ the normal subgroup generated by these elements will be contained within the group

$$\left\{ g : \exists\, a_1, \dots, a_n \in \mathbb{Z}\, \theta(g) = \theta(g_1)^{a_1} \cdots \theta(g_n)^{a_n} \in \mathrm{stQ}^{\mathrm{V}} \right\},$$

where multiplication in $\mathrm{stQ}^{\mathrm{V}}$ (on the right hand side) is defined pointwise by value (cf. section 10.3). However, since there is no reason to assume that anything like theorem 11.5.4 should hold, it seems unlikely that we would have these sets being equivalent to the normal subgroups of G_{v}. Nonetheless, it is clear that in general a normal or closed normal subgroup of G will not be a normal or closed normal subgroup of G_{v}, nor will the reverse necessarily be the case.

Many of the results in this thesis concerned themselves with divisible ordered abelian groups in conjunction with the induced residue map $\widetilde{\varrho}$. A further generalisation may therefore be considered whereby divisible ordered abelian groups are looked at in conjunction with other types of 'residue' map. In particular, alternatives to the inverse limit $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}_n$ might be looked at with $\widehat{\mathbb{Z}}$ replaced by some other profinite group satisfying minimal conditions. If this were generalised to arbitrary divisible ordered abelian groups it may be possible to work backwards in order to establish a 'Presburger like' structure which when factored by a relevant quotient group will map onto these divisible ordered abelian groups.

Chapter 14

Appendix

14.1 Embeddings

This section gives details and proofs relating to the comments found in section 8.2 concerning embeddings and the Hahn product.

Definition 14.1.1. Given $a \in \widetilde{\Gamma}$, we define Γ_a and Γ^a to be as follows:

$$\Gamma_a = \left\{ b \in \widetilde{\Gamma} : \mathrm{st}\left(\frac{b}{a}\right) = 0 \right\}, \quad \Gamma^a = \left\{ b \in \widetilde{\Gamma} : \mathrm{st}\left(\frac{b}{a}\right) \in \mathbb{R} \right\}.$$

Lemma 14.1.2. For all $a \in \mathrm{V}$ there is a map

$$\theta_a \colon \Gamma^a/\Gamma_a \to \Gamma^a$$

with the properties that

1. $\theta_a(\Gamma_a + x) \in \Gamma_a + x$ for all $x \in \Gamma^a$;

2. θ_a is a homomorphism of groups preserving order.

In fact the map θ_a is a choice function of the partition of Γ^a by the cosets of Γ_a.

Proof. We prove this using Zorn's lemma applied to the poset

$$\mathscr{P} = \left\{ (H, \theta) : \begin{array}{l} H \leq \Gamma^a/\Gamma_a \text{ as a } \mathbb{Q}\text{-vector subspace,} \\ \\ \theta\colon H \to \Gamma_a \text{ is a } \mathbb{Q}\text{-linear transformation} \\ \qquad \text{satisfying } \theta(\Gamma_a + x) \in \Gamma_a + x \text{ for all } x \in \Gamma^a \end{array} \right\},$$

with the partial order defined by inclusion in the normal way. It is straightforward to show that all chains have an upper bound, it being the union of the elements of the chain. We may therefore suppose that $\mathscr{P}$ contains a maximal element (H, θ), where $H \leq \Gamma^a/\Gamma_a$ is a $\mathbb{Q}$-vector subspace, and $\theta\colon H \to \Gamma^a$ a $\mathbb{Q}$-linear transformation satisfying $\theta(\Gamma_a + x) \in \Gamma_a + x$ for all $x \in \Gamma^a$. In order to show that $H = \Gamma^a/\Gamma_a$ we suppose otherwise and derive a contradiction.

If $H \neq \Gamma^a/\Gamma_a$ then there exists some $x \in \Gamma^a$ such that $\Gamma_a + x \notin H$. We extend θ to $\theta'\colon \langle H \cup \{\Gamma_a + x\}\rangle \to \Gamma^a$ by defining

$$\theta'(\Gamma_a + y + qx) = \theta(\Gamma_a + y) + qx \quad \text{for} \quad y \in H, q \in \mathbb{Q}.$$

If $\Gamma_a + y + qx = \Gamma_a + y' + q'x$ then $\Gamma_a + (q - q')x = \Gamma_a + (y' - y) \in H$ so $q = q'$ (otherwise we would have $\Gamma_a + x = \frac{1}{q-q'}(\Gamma_a + (y' - y)) \in H$). Also, $\Gamma_a + y = \Gamma_a + y'$, hence $\theta(\Gamma_a + y) = \theta(\Gamma_a + y')$ and θ' is well-defined.

This θ' obviously has the properties required, contradicting the fact that H was maximal. Hence $H = \Gamma^a/\Gamma_a$ which gives us the result.

□

Proposition 14.1.3. The map $\bigoplus_{a \in \mathrm{V}} \Gamma^a/\Gamma_a \to \widetilde{\Gamma}$ is an injective map of ordered groups.

Proof. We take

$$(\dots, \Gamma_a + x, \dots) \mapsto \sum_{a \in \mathrm{V}} \theta_a(\Gamma_a + x).$$

We must check that in a sum

$$\sum_a \lambda_a \theta_a(\Gamma_a + x_a) = 0$$

the λ_a are all 0. But for $a < b$ we see that $q|\theta_a(\Gamma_a + x_a)| < |\theta_b(\Gamma_b + x_b)|$ for all $q \in \mathbb{Q}$. So the $\theta_a(\Gamma_a + x_a)$ are all linearly independent in $\widetilde{\Gamma}$ as a vector space over $\mathbb{Q}$. □

Notation. For a linear order $\prec$ on a set S,

1. Let $\prec^*$ denote the reverse ordering. *i.e.* $x \prec y \Longleftrightarrow y \prec^* x$;

2. We say that $\prec$ is a **well-order** if, for any non-empty subset X of S, this subset has a $\prec$-least element.

Definition 14.1.4. For a family $G_i (i \in I)$ of ordered abelian groups, where I is linearly ordered by $<$, define the **Hahn product** $\prod^{\mathrm{H}}_{i\in I} G_i$ of the G_i to be the set of functions $f \colon I \to \cup G_i$ so that $f(i) \in G_i$ for all $i \in I$ and

$$\operatorname{supp}(f) = \{\, i : f(i) \neq 0\}$$

is well-ordered by $<^*$.

The Hahn product is a subset of the standard cartesian product $\prod_{i\in I} G_i$ with addition induced from this. As can be seen from the definition, the difference is precisely that if $(g_i)_{i\in I} \in \prod_{i\in I} G_i$ then $(g_i)_{i\in I} \in \prod^{\mathrm{H}}_{i\in I} G_i$ if and only if the set $\{\, g_i : g_i \neq 0, i \in I\}$ is well-ordered. We can define an ordering on $\prod^{\mathrm{H}}_{i\in I} G_i$ as follows:

For $f \neq g$ define $f < g$ if and only if $f(i) < g(i)$ for the $<^*$-least i such that $i \in \operatorname{supp}(f - g)$. Such an i exists since finite unions and subsets of well-ordered sets are well-ordered and clearly $\operatorname{supp}(f - g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$.

Using this ordering we see that $\prod^{\mathrm{H}}_{i\in I} G_i$ is an ordered abelian group. $\prod^{\mathrm{H}}_{i\in I} G_i$ also has a natural valuation to I

$$\mathrm{v}(f) = <^* \text{-least } i \in \operatorname{supp}(f),$$

and this valuation satisfies the properties of proposition 8.2.3.

Notation. In the following theorem where $x \in \widetilde{\Gamma}$ and $\mathrm{v}(x) = a$ we will write $(x)_a$ in the place of $\Gamma_a + x$.

Theorem 14.1.5 (Hahn Embedding Theorem). There is a valuation-preserving embedding

$$\eta \colon \widetilde{\Gamma} \to \prod_{a\in \mathrm{V}}^{\mathrm{H}} \Gamma^a/\Gamma_a$$

of ordered groups such that, for any $x \in \widetilde{\Gamma}$ with value a,

$$(\eta(x))_a = \Gamma_a + x.$$

In other words the diagram

$$\begin{array}{ccc} \bigoplus_a \Gamma^a/\Gamma_a & \longrightarrow & \prod^{\mathrm{H}}_a \Gamma^a/\Gamma_a \\ & \searrow \qquad \nearrow_{\eta} & \\ & \widetilde{\Gamma} & \end{array}$$

commutes where the map $\bigoplus_a \Gamma^a/\Gamma_a \to \widetilde{\Gamma}$ is the natural map as described above.

Proof. We begin by showing that there exists a map ν such that the following map commutes:

$$\begin{array}{ccc} \bigoplus_a \Gamma^a/\Gamma_a & \longrightarrow & \prod_a^{\mathrm{H}} \Gamma^a/\Gamma_a \\ & \searrow \qquad \nearrow_{\nu} & \\ & \widetilde{\Sigma} & \end{array}$$

where $\widetilde{\Sigma} = \operatorname{Im}\left(\bigoplus_{a\in\widetilde{\Gamma}} \Gamma^a/\Gamma_a \to \widetilde{\Gamma}\right)$ for the map defined in proposition 14.1.3. We will define the map $\bigoplus_a \Gamma^a/\Gamma_a \to \prod_a^{\mathrm{H}} \Gamma^a/\Gamma_a$ in the most obvious manner, so that

$$\sum_a \Gamma_a + x_a \mapsto (\dots, \Gamma_a + x_a, \dots).$$

This is well defined since the element $(\dots, \Gamma_a + x_a, \dots)$ is only a finite sum and hence $\operatorname{supp}(\sum_a \Gamma_a + x_a)$ is well-ordered. By proposition 14.1.3 every element of $\widetilde{\Sigma}$ can be written uniquely in the form

$$\sum_{a\in A} x_a$$

where A is some finite subset of V and $x_a \in \widetilde{\Gamma}$ is such that $\theta_a(\Gamma_a + x_a) = x_a$. For an element $\sum_{a\in A} x_a$ of $\widetilde{\Sigma}$ we therefore define $\nu\colon \widetilde{\Sigma} \to \prod_a^{\mathrm{H}} \Gamma^a/\Gamma_a$ to map

$$\nu\colon \sum_{a\in A} x_a \mapsto \sum_{a\in A} \Gamma_a + x_a,$$

which is well defined since A is finite, giving $\operatorname{supp}(\sum_{a\in A} \Gamma_a + x_a)$ to be finite. Now it is clear that the diagram commutes since for an element $(\dots, \Gamma_a + x_a, \dots)$ we know that

$$\sum_{a\in A} \Gamma_a + x_a = \sum_{a\in A} \Gamma_a + \theta_a(\Gamma_a + x_a).$$

So given that such a ν exists, we may apply Zorn's lemma to the poset

$$\mathscr{P} = \left\{ (H,\eta) : \begin{array}{l} H \leq \widetilde{\Gamma} \text{ is a } \mathbb{Q}\text{-vector subspace,} \\[2em] H \geq \operatorname{Im}\left(\bigoplus_a \Gamma^a/\Gamma_a \to \widetilde{\Gamma}\right) \\[2em] \eta\colon H \to \prod_{a\in \mathrm{V}}^{\mathrm{H}} \Gamma^a/\Gamma_a \text{ is a valuation preserving} \\ \qquad \text{embedding extending } \nu. \end{array} \right\},$$

with the partial order defined by inclusion in the normal way. It is straightforward to show that all chains have an upper bound, it being the union of the elements of the chain. We also note that since the map $\bigoplus_a \Gamma^a/\Gamma_a \to \widetilde{\Gamma}$ as defined in proposition 14.1.3 is an injective map of ordered groups the image of this map is a $\mathbb{Q}$-vector subspace of Γ. Hence $\left(\mathrm{Im}\left(\bigoplus_a \Gamma^a/\Gamma_a \to \widetilde{\Gamma}\right), \nu\right) \in \mathscr{P}$ ensuring that $\mathscr{P} \neq \emptyset$. We may therefore suppose that $\mathscr{P}$ contains a maximal element (H, η). We suppose that $H \neq \widetilde{\Gamma}$ and that η cannot be extended in order to establish a contradiction.

Let η be a map $\eta \colon H \to \prod_a^{\mathrm{H}} \Gamma^a/\Gamma_a$ where $H < \widetilde{\Gamma}$ is a vector subspace containing $\mathrm{Im}\left(\bigoplus_a \Gamma^a/\Gamma_a \to \widetilde{\Gamma}\right)$. We use transfinite induction to construct sequences of values $a_i \in \mathrm{V}$ indexed by ordinals i, and show that these sequences can be extended indefinitely. This will contradict the fact that the set of values is bounded in size by the cardinality of $\widetilde{\Gamma}$.

Suppose α is some ordinal. Then for our inductive hypothesis we suppose that we have the following:-

A decreasing sequence $a_j < a_i$ for $j < i$ of elements $a_i \in \mathrm{V}$ for $i < \alpha$;
A sequence of elements $\gamma_i \in \widetilde{\Gamma} \setminus H$ for $i < \alpha$ where i not a limit ordinal;
A sequence of elements $\delta_i \in H$ for $i < \alpha$;

such that:

$a_i = \mathrm{v}(\gamma_i)$ for $i < \alpha$
$\gamma_i = \gamma_0 - \delta_i$ for $i < \alpha$
$\eta \colon \delta_i \mapsto \left(\sum_{j<i} \Gamma_{a_j} + \gamma_j\right) + \epsilon$ where $\mathrm{v}(\epsilon) \leq a_i$.

At the initial step where $\alpha = 1$ set $\gamma_0 = \gamma, \delta_0 = 0$ and $a_0 = \mathrm{v}(\gamma_0)$. There is nothing to prove for the initial step.

We now consider the inductive step for successor ordinals. So suppose we have sequences satisfying our inductive hypothesis for some ordinal α. Then we have

$$a_0, \dots, a_\alpha$$
$$\gamma_0, \dots, \gamma_\alpha$$
$$\delta_0, \dots, \delta_\alpha$$

and we wish to find some $\delta_{\alpha+1}$ which satisfies our criteria and which will also allow us to find $a_{\alpha+1}$ and $\gamma_{\alpha+1}$ of the required sort. So consider the map

$$\eta^* \colon \langle H, \gamma_0 \rangle \to \prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a/\Gamma_a$$

which maps

$$\eta^* \colon \gamma_0 \mapsto \sum_{j \leq \alpha} \Gamma_{a_j} + \gamma_j$$

and which extends η linearly.

We claim that in this situation, η^* would be an extension of η satisfying our Zorn's lemma requirements (and hence we derive a contradiction). Clearly $H' = \langle H, \gamma_0 \rangle \leq \widetilde{\Gamma}$ and $H \geq \operatorname{Im}\left(\bigoplus_a \Gamma^a/\Gamma_a \to \widetilde{\Gamma}\right)$. Moreover it is clear that η^* extends ν. We must show that η^* preserves values and is an embedding.

Suppose that η^* does not preserve values. Then for some $h'' \in H$ and some $q \in \mathbb{Q} \setminus \{0\}$ we have

$$\mathrm{v}(q\gamma_0 - h'') \neq \mathrm{v}(\eta^*(q\gamma_0 - h'')).$$

Let $h = q^{-1}h''$. Then the above is equivalent to the statement

$$\mathrm{v}(\gamma_0 - h) \neq \mathrm{v}(\eta^*(\gamma_0 - h))$$

which we can write as

$$\mathrm{v}(\gamma_0 - h) \neq \mathrm{v}\left(\left(\sum_{j\leq\alpha} \Gamma_{a_j} + \gamma_j\right) - \eta(h)\right). \tag{14.1}$$

Now $\mathrm{v}(\gamma_0) = a_0$ and $\mathrm{v}(\sum_{j\leq\alpha} \Gamma_{a_j} + \gamma_j) = a_0$. Suppose $\mathrm{v}(h) < a_0$. Then by proposition 8.2.3 (part 4) we know that $\mathrm{v}(\gamma_0 - h) = a_0$. But η preserves values, so $\mathrm{v}(\eta(h)) < a_0$ and hence $\mathrm{v}((\sum_{j\leq\alpha} \Gamma_{a_j} + \gamma_j) - \eta(h)) = a_0$ as well, contradicting equation 14.1. Similarly if $\mathrm{v}(h) > a_0$ we find that

$$\mathrm{v}(\gamma_0 - h) = \mathrm{v}(h) = \mathrm{v}\left(\left(\sum_{j\leq\alpha} \Gamma_{a_j} + \gamma_j\right) - \eta(h)\right),$$

which contradicts equation 14.1. We therefore have that $\mathrm{v}(\gamma_0) = \mathrm{v}(h) = a_0$.

Now suppose $\mathrm{v}(\gamma_0 - h) = a_0$. Then $\gamma_0 - h \notin \Gamma_{a_0}$. Then $(\eta(h))_{a_0} = \Gamma_{a_0} + h$ since η satisfies our Zorn's lemma requirements and so $\Gamma_{a_0} + \gamma_0 - \eta(h) = \Gamma_{a_0} + \gamma_0 - h \neq \Gamma_{a_0}$. But then $\mathrm{v}\left(\left(\sum_{j\leq\alpha} \Gamma_{a_j} + \gamma_j\right) - \eta(h)\right) = a_0$ which again contradicts equation 14.1. We must therefore assume that $\mathrm{v}(\gamma_0 - h) < a_0$.

There are now three cases to consider:

1. $\mathrm{v}(\gamma_0 - h) > a_\alpha$;

2. $\mathrm{v}(\gamma_0 - h) = a_\alpha$;

3. $\mathrm{v}(\gamma_0 - h) < a_\alpha$.

We consider each of them in turn.

So suppose that $\mathrm{v}(\gamma_0 - h) > a_\alpha$. Setting $a = \mathrm{v}(\gamma_0 - h)$ we have $a_\alpha < a < a_0$. By the well-ordering of the a_i's there is some ordinal s such that a_s is the largest value smaller than or equal to a. Moreover by our inductive hypothesis, we know that

$$\eta \colon \delta_{s+1} \mapsto \Big(\sum_{j<s+1} \Gamma_{a_j} + \gamma_j\Big) + \epsilon,$$

where $\mathrm{v}(\epsilon) \leq a_{s+1}$. We also know that $\mathrm{v}(\gamma_0 - \delta_{s+1}) = a_{s+1} < a$. So $\mathrm{v}(h - \delta_{s+1}) = \mathrm{v}(\gamma_0 - \gamma_0 - (h - \delta_{s+1})) = \mathrm{v}((\gamma_0 - h) - (\gamma_0 - \delta_{s+1})) = a$. Also, since $\delta_{s+1} \in H$ and η preserves values, we have that $\mathrm{v}(\eta(h - \delta_{s+1})) = a$. Also

$$\begin{aligned}
\mathrm{v}(\eta^*(\gamma_0 - h)) &= \mathrm{v}(\eta^*(\gamma_0) - \eta(h)) \\
&= \mathrm{v}\Big(\Big(\sum_{j<s+1} \Gamma_{a_j} + \gamma_j\Big) + \Big(\sum_{s+1 \leq j \leq \alpha} \Gamma_{a_j} + \gamma_j\Big) - \eta(h - \delta_{s+1} + \delta_{s+1})\Big) \\
&= \mathrm{v}\Big(\Big(\sum_{s+1 \leq j \leq \alpha} \Gamma_{a_j} + \gamma_j\Big) - \eta(h - \delta_{s+1})\Big) \\
&= a
\end{aligned}$$

since

$$\mathrm{v}\Big(\sum_{s+1 \leq j \leq \alpha} \Gamma_{a_j} + \gamma_j\Big) < a.$$

This contradicts equation 14.1.

In the second case we have $\mathrm{v}(\gamma_0 - h) = a_\alpha$. We know that α is a successor ordinal, so $\alpha - 1$ is well defined. We also know by our inductive hypothesis that $\gamma_\alpha = \gamma_0 - \delta_\alpha$. Substituting this into $\mathrm{v}(\gamma_0 - h) = a_\alpha$ we get $\mathrm{v}(\gamma_\alpha - (h - \delta_\alpha)) = a_\alpha$ and hence $\mathrm{v}(h - \delta_\alpha) \leq a_\alpha$ since $\mathrm{v}(\gamma_\alpha) = a_\alpha$. Now

$$\begin{aligned}
\mathrm{v}(\eta^*(\gamma_0 - h)) &= \mathrm{v}\Big(\Big(\sum_{j \leq \alpha - 1} \Gamma_{a_j} + \gamma_j\Big) + \Gamma_{a_\alpha} + \gamma_\alpha - \eta(h - \delta_\alpha + \delta_\alpha)\Big) \\
&= \mathrm{v}(\Gamma_{a_\alpha} + \gamma_\alpha - \eta(h - \delta_\alpha))
\end{aligned} \tag{14.2}$$

and so $\mathrm{v}(\eta(h - \delta_\alpha)) = a_\alpha$. Since η satisfies our Zorn's lemma requirements we know from this that $(\eta(h - \delta_\alpha))_{a_\alpha} = \Gamma_{a_\alpha} + h - \delta_\alpha$. Now $\gamma_0 - h \notin \Gamma_{a_\alpha}$ since $\mathrm{v}(\gamma_0 - h) = a_\alpha$ so $\gamma_\alpha - h + \delta_\alpha \notin \Gamma_{a_\alpha}$ and so $\gamma_\alpha - \eta(h - \delta_\alpha) \notin \Gamma_{a_\alpha}$. From this and equation 14.2 it then follows that $\mathrm{v}(\eta^*(\gamma_0 - h)) = a_\alpha$, which contradicts our assumption 14.1.

We conclude from this that the third possibility must be the case. In other words, $\mathrm{v}(\gamma_0 - h) < a_\alpha$. But in this case we claim that

$$\eta \colon h \mapsto \Big(\sum_{j \leq \alpha} \Gamma_{a_j} + \gamma_j\Big) + \epsilon$$

where $\mathrm{v}(\epsilon) < a_\alpha$. To prove this, take any $j \leq \alpha$. Then

$$\begin{aligned}\mathrm{v}(h - \delta_j) &= \mathrm{v}((\gamma_0 - \delta_j) - (\gamma_0 - h)) \\ &= \mathrm{v}(\gamma_j - (\gamma_0 - h)) \\ &= a_j.\end{aligned}$$

So $(\eta(h - \delta_j))_{a_j} = \Gamma_{a_j} + h - \delta_j$ for all $j \leq \alpha$ since η satisfies our Zorn's lemma requirements. But then

$$\begin{aligned}(\eta^*(\gamma_0 - h))_{a_j} &= (\eta^*((\gamma_0 - \delta_j) - (h - \delta_j)))_{a_j} \\ &= \Gamma_{a_j} + \gamma_j - (h - \delta_j) \\ &= \Gamma_{a_j} + \gamma_0 - h \\ &= \Gamma_{a_j}.\end{aligned}$$

So $(\eta(h))_{a_j} = (\eta^*(\gamma_0))_{a_j} = \Gamma_{a_j} + \gamma_j$ for all $j \leq \alpha$, from which we may conclude that

$$\eta\colon h \mapsto \Big(\sum_{j \leq \alpha} \Gamma_{a_j} + \gamma_j\Big) + \epsilon$$

where $\mathrm{v}(\epsilon) < a_\alpha$ as required. Since we are assuming that $\mathrm{v}(\gamma_0 - h) < a_\alpha$ we can set $\delta_{\alpha+1} = h$ in order to complete the inductive step.

In all of the above cases we either have η^* preserving values, or we can continue to the next inductive step. However, if we suppose that η^* preserves values we also have that η^* is injective. For suppose η^* is not injective. Then $\eta^*(h + q\gamma_0) = 0$ for some $h \in H, q \in \mathbb{Q} \setminus \{0\}$. But then $\eta^*(q^{-1}h + \gamma_0) = 0$ and η^* preserves values, so $\mathrm{v}(q^{-1}h + \gamma_0) = 0$. In fact, this tells us that $q^{-1}h = \gamma_0$ which contradicts the fact that $q^{-1}h \in H$ whilst $\gamma_0 \notin H$. The map η^* must therefore be injective. But then η^* satisfies all of our Zorn's lemma requirements, which contradicts the maximality of (H, η). The only possibility therefore is that the induction continues.

We now consider the inductive step for limit ordinals. Suppose we have sequences satisfying our inductive hypothesis for some limit ordinal λ. Then we have

$$\begin{gathered}a_0, \ldots, a_i, \ldots \\ \gamma_0, \ldots, \gamma_i, \ldots \\ \delta_0, \ldots, \delta_i, \ldots\end{gathered}$$

for $i < \lambda$. We wish to find some δ_λ which satisfies our criteria and which will also allow us to find a_λ and γ_λ of the required sort. So consider the map

$$\eta^*\colon \langle H, \gamma \rangle \to \prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a / \Gamma_a$$

which maps

$$\eta^*\colon \gamma_0 \mapsto \sum_{j<\alpha} \Gamma_{a_j} + \gamma_j$$

and which extends η linearly. We note that this element which we are mapping γ to is contained in the Hahn product since the a_i are indexed by ordinals; hence its support is well-ordered. We claim that η^* preserves values. Suppose otherwise, then as for the successor stage we can find some $h \in H$ such that

$$\mathrm{v}(\gamma_0 - h) \neq \mathrm{v}\left(\left(\sum_{j<\alpha}\Gamma_{a_j} + \gamma_j\right) - \eta(h)\right). \tag{14.3}$$

Similar arguments to those given above will show that $\mathrm{v}(\gamma_0) = \mathrm{v}(h) = a_0$ and that $\mathrm{v}(\gamma_0 - h) < a_j$ for all $j < \alpha$. But in this case we claim that

$$\eta\colon h \mapsto \left(\sum_{j<\alpha}\Gamma_{a_j} + \gamma_j\right) + \epsilon$$

where $\mathrm{v}(\epsilon) < a_j$ for every $j < \alpha$. To prove this take any $j < \alpha$. Since η preserves values we know that

$$\begin{aligned}
\mathrm{v}(\eta(h - \delta_{j+1})) &= \mathrm{v}(h - \delta_{j+1}) \\
&= \mathrm{v}((\gamma_0 - \delta_{j+1}) - (\gamma_0 - h)) \\
&= \mathrm{v}(\gamma_{j+1} - (\gamma_0 - h)).
\end{aligned}$$

But $\mathrm{v}(\gamma_{j+1}) = a_{j+1} < a_j$ and $\mathrm{v}(\gamma_0 - h) < a_j$ so

$$\mathrm{v}(\eta(h - \delta_{j+1})) < a_j. \tag{14.4}$$

Also

$$\begin{aligned}
\eta^*(\gamma_0 - h) &= \eta^*((\gamma_0 - \delta_{j+1}) - (h - \delta_{j+1})) \\
&= \left(\sum_{j+1\leq i<\alpha} \Gamma_{a_i} + \gamma_i\right) - \eta(h - \delta_{j+1}),
\end{aligned}$$

and since $\mathrm{v}\left(\sum_{j+1\leq i<\alpha}\Gamma_{a_i} + \gamma_i\right) = a_{j+1} < a_j$ it follows from this and equation 14.4 that $\mathrm{v}(\eta^*(\gamma_0 - h)) < a_j$ for all $j < \alpha$. What we conclude from this is that

$$(\eta^*(\gamma_0 - h))_{a_j} = \Gamma_{a_j},$$

so $\Gamma_{a_j} + \gamma_j - (\eta(h))_{a_j} = \Gamma_{a_j}$ from which $(\eta(h))_{a_j} = \Gamma_{a_j} + \gamma_j$ and hence

$$\eta\colon h \mapsto \left(\sum_{j<\alpha}\Gamma_{a_j} + \gamma_j\right) + \epsilon$$

where $\mathrm{v}(\epsilon) < a_j$ for every $j < \alpha$ as required. Since we are assuming that $\mathrm{v}(\gamma_0 - h) < a_j$ for every $j < \alpha$ we can set $\delta_\alpha = h$ in order to complete the inductive step.

As with the successor stage, in every event either η^* preserves values, or we can continue to the next inductive step. However, if we suppose that η^* preserves values we also have that η^* is injective for exactly the same reason given for the successor step. But then η^* satisfies all of our Zorn's lemma requirements, which contradicts the maximality of (H, η). The only possibility therefore is that the induction continues.

This completes the induction. We therefore find that we can extend the sequences $(a_i), (\gamma_i)$ and (δ_i) indefinitely, which in particular contradicts the fact that the set of values is bounded in size by the cardinality of $\widetilde{\Gamma}$. We must conclude therefore that $H = \widetilde{\Gamma}$ is the maximal element in our poset $\mathscr{P}$. $\square$

Proposition 14.1.6.[1] There exists an ordering on $\prod_{a \in \mathrm{V}} \Gamma^a/\Gamma_a$ extending the ordering on the subgroup

$$\prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a/\Gamma_a.$$

Proof. Both $\prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a/\Gamma_a$ and $\prod_{a \in \mathrm{V}} \Gamma^a/\Gamma_a$ can be taken as vector spaces over $\mathbb{Q}$. We therefore produce an embedding of ordered vector spaces which should suffice. Consider the vector space

$$V = \prod_{a \in \mathrm{V}} \Gamma^a/\Gamma_a \Big/ \prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a/\Gamma_a \ ,$$

and let B be a basis for V. We can order V by defining a well-order on the basis B. Let f be a choice function $f \colon B \to \prod_{a \in \mathrm{V}} \Gamma^a/\Gamma_a$ so that $f(b) \in \prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a/\Gamma_a + b$ for all $b \in B$. We can extend $f(B)$ to a basis for $\prod_{a \in \mathrm{V}} \Gamma^a/\Gamma_a$ by taking elements only from $\prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a/\Gamma_a$. We then define an ordering on $\prod_{a \in \mathrm{V}} \Gamma^a/\Gamma_a$ in the following way:

Using the basis extending $f(B)$ we see that for any $a, b \in \prod_{a \in \mathrm{V}} \Gamma^a/\Gamma_a$ there exist a pair of unique sums

$$a = q_1 f(b_1) + \cdots + q_n f(b_n) + s$$

and

$$b = q'_1 f(b'_1) + \cdots + q'_m f(b'_m) + s'$$

with $q_1, \ldots, q_n, q'_1, \ldots, q'_m \in \mathbb{Q}$ and so that $b_1, \ldots, b_n, b'_1, \ldots, b'_m \in B$ and $s, s' \in \prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a/\Gamma_a$.

[1] Many thanks to John Graham on whose argument the proof for this proposition is based.

We then set $a < b$ if

$$q_1 b_1 + \cdots + q_n b_n < q_1' b_1' + \cdots + q_m' b_m'$$

or

$$q_1 b_1 + \cdots + q_n b_n = q_1' b_1' + \cdots + q_m' b_m' \quad \text{and} \quad s < s' \text{ in } \prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a / \Gamma_a.$$

We then clearly have the identity map being a natural embedding

$$\prod_{a \in \mathrm{V}}^{\mathrm{H}} \Gamma^a / \Gamma_a \to \prod_{a \in \mathrm{V}} \Gamma^a / \Gamma_a$$

of ordered $\mathbb{Q}$-vector spaces (and hence ordered abelian groups). □

Bibliography

[1] Roger Alperin and Hyman Bass. Length functions of group actions on Λ-trees. In *Combinatorial group theory and topology (Alta, Utah, 1984)*, pages 265–378. Princeton Univ. Press, Princeton, NJ, 1987.

[2] Sergei Artëmov and Franco Montagna. On first-order theories with provability operator. *J. Symbolic Logic*, 59(4):1139–1153, 1994.

[3] Nakhlé H. Asmar and Stephen J. Montgomery-Smith. Hahn's embedding theorem for orders and harmonic analysis on groups with ordered duals. *Colloq. Math.*, 70(2):235–252, 1996.

[4] James Ax and Simon Kochen. Diophantine problems over local fields. I. *Amer. J. Math.*, 87:605–630, 1965.

[5] James Ax and Simon Kochen. Diophantine problems over local fields. II. A complete set of axioms for p-adic number theory. *Amer. J. Math.*, 87:631–648, 1965.

[6] B. Barras, S. Boutin, C. Cornes, J. Courant, J.C. Filliatre, E. Giménez, H. Herbelin, G. Huet, H. Laulhère, C. Mu noz, C. Murthy, C. Parent, P. Loiseleur, C. Paulin, A. Saïbi, and B. Werner. The Coq Proof Assistant Reference Manual – Version V6.3.1. Technical report, INRIA, December 1999.

[7] E. Börger, H. Kleine Büning, M. M. Richter, and W. Schönfeld, editors. *Computer science logic*, Berlin, 1991. Springer-Verlag.

[8] Nicolas Bourbaki. *Elements of mathematics. Theory of sets.* Hermann, Publishers in Arts and Science, Paris, 1968. Translated from the French.

[9] J. Richard Büchi. Regular canonical systems. *Arch. Math. Logik Grundlagenforsch.*, 6:91–111 (1964), 1964.

[10] Ian M. Chiswell. Nontrivial group actions on Λ-trees. *Bull. London Math. Soc.*, 24(3):277–280, 1992.

[11] Ian M. Chiswell. Some examples of groups with no nontrivial action on a Λ-tree. *Mathematika*, 42(1):214–219, 1995.

[12] Ian M. Chiswell. Minimal group actions on Λ-trees. *Proc. Roy. Soc. Edinburgh Sect. A*, 128(1):23–36, 1998.

[13] Paul Conrad, John Harvey, and Charles Holland. The Hahn embedding theorem for abelian lattice-ordered groups. *Trans. Amer. Math. Soc.*, 108:143–169, 1963.

[14] D.C. Cooper. Theorem proving in arithmetic without multiplication. In B. Meltzer and D. Michie, editors, *Machine intelligence 7*, pages 91–99. Edinburgh University Press, 1972.

[15] Françoise Delon. Model theory of Henselian valued fields. In *Logic Colloquium '87 (Granada, 1987)*, pages 1–10. North-Holland, Amsterdam, 1989.

[16] Herbert B. Enderton. *Elements of set theory*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1977.

[17] Ju. L. Ershov. On the elementary theory of maximal normed fields. *Dokl. Akad. Nauk SSSR*, 165:21–23, 1965.

[18] Michael J. Fischer and Michael O. Rabin. Super-exponential complexity of Presburger arithmetic. pages 27–41. SIAM–AMS Proc., Vol. VII, 1974.

[19] Isidore Fleischer. The Hahn embedding theorem: analysis, refinements, proof. In *Algebra, Carbondale 1980 (Proc. Conf., Southern Illinois Univ., Carbondale, Ill., 1980)*, pages 278–290. Springer, Berlin, 1981.

[20] L. Fuchs. *Partially ordered algebraic systems*. Pergamon Press, Oxford, 1963.

[21] Steven R. Givant. A portrait of Alfred Tarski. *Math. Intelligencer*, 13(3):16–32, 1991.

[22] A.M.W. Glass. *Partially ordered groups*. World Scientific Publishing Co., 1999.

[23] Erich Grädel. Dominoes and the complexity of subclasses of logical theories. *Ann. Pure Appl. Logic*, 43(1):1–30, 1989.

[24] K. A. H. Gravett. Valued linear spaces. *Quart. J. Math., Oxford Ser. (2)*, 6:309–315, 1955.

[25] H. Hahn. Über die nichtarchimedischen Grössensysteme. *S.-B. Kaiserlichen Akad. Wiss. Math. Nat. Kl.*, IIa:601–665, 1907.

[26] Joseph Y. Halpern. Presburger arithmetic with unary predicates is Π^1_1 complete. *J. Symbolic Logic*, 56(2):637–642, 1991.

[27] Victor Harnik. ω_1-like recursively saturated models of Presburger's arithmetic. *J. Symbolic Logic*, 51(2):421–429, 1986.

[28] C. Hosono and Y. Ikeda. A formal derivation of the decidability of the theory SA. *Theoret. Comput. Sci.*, 127(1):1–23, 1994.

[29] Richard Kaye. *Models of Peano Arithmetic.* Oxford University Press, Oxford, 1991.

[30] Richard Kaye. Infinitary definitions of equivalence relations in models of PA. *Ann. Pure Appl. Logic*, 89(1):37–43, 1997. Logic Colloquium '94 (Clermont-Ferrand).

[31] Richard Kaye. Presburger arithmetic. Notes from a Birmingham University study group, 1997.

[32] Richard Kaye and Dugald Macpherson, editors. *Automorphisms of first-order structures.* The Clarendon Press Oxford University Press, New York, 1994.

[33] H.J. Keisler and C.C Chang. *Model Theory.* North-Holland Publishing Company, Amsterdam, 1973.

[34] I. Korec. Decidable and undecidable theories of generalized Pascal triangles. In *General algebra and discrete mathematics (Potsdam, 1993)*, pages 169–179. Heldermann, Lemgo, 1995.

[35] Roman Kossak and Henryk Kotlarski. Results on automorphisms of recursively saturated models of PA. *Fund. Math.*, 129(1):9–15, 1988.

[36] Roman Kossak and James H. Schmerl. The automorphism group of an arithmetically saturated model of Peano arithmetic. *J. London Math. Soc. (2)*, 52(2):235–244, 1995.

[37] Henryk Kotlarski. On elementary cuts in models of arithmetic. *Fund. Math.*, 115(1):27–31, 1983.

[38] Henryk Kotlarski. On elementary cuts in recursively saturated models of Peano arithmetic. *Fund. Math.*, 120(3):205–222, 1984.

[39] Christian Michaux and Roger Villemaire. Cobham's theorem seen through Büchi's theorem. In *Automata, languages and programming (Lund, 1993)*, pages 325–334. Springer, Berlin, 1993.

[40] Christian Michaux and Roger Villemaire. Open questions around Büchi and Presburger arithmetics. In *Logic: from foundations to applications (Staffordshire, 1993)*, pages 353–383. Oxford Univ. Press, New York, 1996.

[41] Christian Michaux and Roger Villemaire. Presburger arithmetic and recognizability of sets of natural numbers by automata: new proofs of Cobham's and Semenov's theorems. *Ann. Pure Appl. Logic*, 77(3):251–277, 1996.

[42] Sumio Morioka, Naoki Shibata, Teruo Higashino, and Kenichi Taniguchi. Techniques to reduce computation time in decision procedure for prenex normal form Presburger sentences bounded only by existential quantifiers. *Trans. Inform. Process. Soc. Japan*, 38(12):2419–2426, 1997.

[43] Roman Murawski. Contributions of Polish logicians to decidability theory. *Modern Logic*, 6(1):37–66, 1996.

[44] Hirosi Nagao and Yukio Tsushima. *Representations of finite groups.* Academic Press Inc., Boston, MA, 1989. Translated from the Japanese.

[45] Derek C. Oppen. A superexponential upper bound on the complexity of Presburger arithmetic. *J. Comput. System Sci.*, 16(3):323–332, 1978.

[46] Mojżesz Presburger. On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation. *Hist. Philos. Logic*, 12(2):225–233, 1991. Translated from the German and with commentaries by Dale Jacquette.

[47] M. A. Rincón and L. M. R. Gadelha Jr. Some applications of (semi-)decision algorithms for presburger arithmetic in automated deduction based on rewriting

techniques. *La Revista de la Sociedad Chilena de Ciencia de la Computacion*, 2(1):14–23, 1997.

[48] Abraham Robinson. *Complete theories.* North-Holland Publishing Co., Amsterdam, 1956.

[49] Joseph J. Rotman. *The theory of groups. An introduction.* Allyn and Bacon Inc., Boston, Mass., second edition, 1973. Allyn and Bacon Series in Advanced Mathematics.

[50] U. Schöning. Complexity of Presburger arithmetic with fixed quantifier dimension. *Theory Comput. Syst.*, 30(4):423–428, 1997.

[51] Craig Smoryński. Elementary extensions of recursively saturated models of arithmetic. *Notre Dame J. Formal Logic*, 22(3):193–203, 1981.

[52] Craig Smoryński. *Logical number theory. I.* Springer-Verlag, Berlin, 1991. An introduction.

[53] Lou van den Dries. Quantifier elimination for linear formulas over ordered and valued fields. In *Proceedings of the Model Theory Meeting (Univ. Brussels, Brussels/Univ. Mons, Mons, 1980)*, volume 33, pages 19–31, 1981.

[54] Volker Weispfenning. On the elementary theory of Hensel fields. *Ann. Math. Logic*, 10(1):59–93, 1976.

[55] Volker Weispfenning. Quantifier elimination and decision procedures for valued fields. In *Models and sets (Aachen, 1983)*, pages 419–472. Springer, Berlin, 1984.

[56] Volker Weispfenning. The complexity of almost linear Diophantine problems. *J. Symbolic Comput.*, 10(5):395–403, 1990.

[57] Jan Woleński. Genesis of the Warsaw school of logic. In *Mathematics at the turn of the twentieth century (Polish) (Jaworze, 1988)*, pages 26–34. Uniw. Ślaski, Katowice, 1992.

[58] Jan Zygmunt. Mojżesz Presburger: life and work. *Hist. Philos. Logic*, 12(2):211–223, 1991.